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## Spectral action on $SU_q(2)$

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### Abstract

The spectral action on the equivariant real spectral triple over  $\mathcal{A}(SU_q(2))$  is computed explicitly. Properties of the differential calculus arising from the Dirac operator are studied and the results are compared to the commutative case of the sphere  $\mathbb{S}^3$ .

PACS numbers: 11.10.Nx, 02.30.Sa, 11.15.Kc  
 MSC–2000 classes: 46H35, 46L52, 58B34  
 CPT-P06-2007

<sup>1</sup> UMR 6207

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<sup>4</sup> Partially supported by Polish Government grants 189/6.PRUE/2007/7; 115/E-343/SPB/6.PR UE/DIE and N 201 1770 33

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## 1 Introduction

The quantum group  $SU_q(2)$  has already a rather long history of studies [33] being one of the finest examples of quantum deformation. This includes an approach via the noncommutative notion of spectral triple introduced by Connes [10, 15] and various notions of Dirac operators were introduced in [2, 4, 6, 13, 28]. Finally, a real spectral triple, which was exhibited in [21], is invariant by left and right action of  $\mathcal{U}_q(su(2))$  and satisfies almost all postulated axioms of triples except the commutant and first-order properties. These, however, remain valid only up to infinitesimal of arbitrary high order. The last presentation generalizes in a straightforward way all geometric construction details of the spinorial spectral triple for the classical three-sphere. In

particular, both the equivariant representation and the symmetries have a  $q \rightarrow 1$  proper classical limit.

The goal of this article is to obtain the spectral action defined in [7] by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) := \text{Tr}(\Phi(\mathcal{D}_A/\Lambda)) \quad (1)$$

where  $\mathcal{D}$  is the Dirac operator,  $A$  is a selfadjoint one-form,  $\mathcal{D}_A = \mathcal{D} + A + JAJ^{-1}$  and  $J$  is the reality operator. Here,  $\Phi$  is any even positive cut-off function which could be replaced by a step function up to some mathematical difficulties investigated in [23]. This means that  $\mathcal{S}$  counts the spectral values of  $|\mathcal{D}_A|$  less than the mass scale  $\Lambda$ . Actually, as shown in [8]

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{0 < k \in Sd^+} \Phi_k \Lambda^k \oint |\mathcal{D}_A|^{-k} + \Phi(0) \zeta_{\mathcal{D}_A}(0) + \mathcal{O}(\Lambda^{-1}), \quad (2)$$

$$\zeta_{\mathcal{D}_A}(0) = \zeta_{\mathcal{D}}(0) + \sum_{k=1}^d \frac{(-1)^k}{k} \oint (A\mathcal{D}^{-1})^k. \quad (3)$$

where  $D_A = \mathcal{D}_A + P_A$ ,  $P_A$  the projection on  $\text{Ker } \mathcal{D}_A$ ,  $\Phi_k = \frac{1}{2} \int_0^\infty \Phi(t) t^{k/2-1} dt$ ,  $d$  is the spectral dimension of the triple and  $Sd^+$  is the strictly positive part of the dimension spectrum  $Sd$  of  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ . Here,  $Sd^+ = Sd = \{1, 2, 3\}$ , so

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = \sum_{1 \leq k \leq 3} \Phi_k \Lambda^k \oint |\mathcal{D}_A|^{-k} + \Phi(0) \zeta_{\mathcal{D}_A}(0). \quad (4)$$

Recall that the tadpole of order  $\Lambda^k$  is the linear term in  $A \in \Omega_D^1(\mathcal{A})$  in the  $\Lambda^k$  part of (4).

Note that there are no terms in  $\Lambda^{-k}$ ,  $k > 0$  because the dimension spectrum is bounded below by 1. This spectral action has been computed on few examples: [3, 8, 9, 15, 22, 24–26, 30, 34].

Here, we compute (4) with the main difficulty which is to control the differential calculus generated by the Dirac operator. To proceed, we introduce two presentations of one-forms. The main ingredient is  $F = \text{sign}(\mathcal{D})$  which appears to be a one-form up to  $OP^{-\infty}$ .

In section 2, we discuss the spectral action of an arbitrary 3-dimensional spectral triple using cocycles.

In sections 3 and 4 we recall the main results on  $SU_q(2)$  of [21] and show that the full spectral action with reality operator given by (4) is completely determined by the terms

$$\oint A^q |\mathcal{D}|^{-p}, \quad 1 \leq q \leq p \leq 3.$$

This question of computation of spectral action was addressed in the epilogue of [37].

In section 5, we establish a differential calculus up to some ideal in pseudodifferential operators and apply these results to the precise computation of previous noncommutative integrals.

Section 6 is devoted to explicit examples, while in next section are given different comparisons with the commutative case of the 3-sphere corresponding to  $SU(2)$ .

## 2 Spectral action in 3-dimension

### 2.1 Tadpole and cocycles

Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  be a spectral triple of dimension 3. For  $n \in \mathbb{N}^*$  and  $a_i \in \mathcal{A}$ , define

$$\phi_n(a_0, \dots, a_n) := \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} \dots [\mathcal{D}, a_n] \mathcal{D}^{-1}.$$

We also use notational integrals on the universal  $n$ -forms  $\Omega_u^n(\mathcal{A})$  defined by

$$\int_{\phi_n} a_0 da_1 \cdots da_n := \phi_n(a_0, a_1, \dots, a_n).$$

and the reordering fact that  $(da_0)a_1 = d(a_0a_1) - a_0da_1$ .

We use the  $b - B$  bicomplex defined in [10]:  $b$  is the Hochschild coboundary map (and  $b'$  is truncated one) defined on  $n$ -cochains  $\phi$  by

$$\begin{aligned} b\phi(a_0, \dots, a_{n+1}) &:= b'\phi(a_0, \dots, a_{n+1}) + (-1)^{n+1}\phi(a_{n+1}a_0, a_1, \dots, a_n), \\ b'\phi(a_0, \dots, a_{n+1}) &:= \sum_{j=0}^n (-1)^j \phi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}). \end{aligned}$$

Recall that  $B_0$  is defined on the normalized cochains  $\phi_n$  by

$$B_0\phi_n(a_0, a_1, \dots, a_{n-1}) := \phi_n(1, a_0, \dots, a_{n-1}), \text{ thus } \int_{\phi_n} d\omega = \int_{B_0\phi_n} \omega \text{ for } \omega \in \Omega_u^{n-1}(\mathcal{A}).$$

Then  $B := NB_0$ , where  $N := 1 + \lambda + \dots \lambda^n$  is the cyclic skewsymmetrizer on the  $n$ -cochains and  $\lambda$  is the cyclic permutation  $\lambda\phi(a_0, \dots, a_n) := (-1)^n \phi(a_n, a_0, \dots, a_{n-1})$ .

We will also encounter the cyclic 1-cochain  $N\phi_1$ :

$$N\phi_1(a_0, a_1) := \phi_1(a_0, a_1) - \phi_1(a_1, a_0) \text{ and } \int_{N\phi_1} a_0 da_1 := N\phi_1(a_0, a_1).$$

**Remark 2.1.** Assume the integrand of  $\mathfrak{f}$  is in  $OP^{-3}$ . Since  $[\mathcal{D}^{-1}, a] = -\mathcal{D}^{-1}[\mathcal{D}, a]\mathcal{D}^{-1} \in OP^{-2}$ , this commutator introduces a integrand in  $OP^{-4}$  so has a vanishing integral: under the integral, we can commute  $\mathcal{D}^{-1}$  with all  $a \in \mathcal{A}$  and all one-forms.

**Lemma 2.2.** We have

- (i)  $b\phi_1 = -\phi_2$ .
- (ii)  $b\phi_2 = 0$ .
- (iii)  $b\phi_3 = 0$ .
- (iv)  $B\phi_1 = 0$ .
- (v)  $B_0\phi_2 = -(1 - \lambda)\phi_1$ .
- (vi)  $bB_0\phi_2 = 2\phi_2 + B_0\phi_3$ .
- (vii)  $B\phi_2 = 0$ .
- (viii)  $B_0\phi_3 = Nb'\phi_1$ .
- (ix)  $B\phi_3 = 3B_0\phi_3$ .

*Proof.* (i)

$$\begin{aligned} b\phi_1(a_0, a_1, a_2) &= \oint a_0 a_1 [\mathcal{D}, a_2] \mathcal{D}^{-1} - \oint a_0 (a_1 [\mathcal{D}, a_2] + [\mathcal{D}, a_1] a_2) \mathcal{D}^{-1} + \oint a_2 a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} \\ &= \oint a_0 [\mathcal{D}, a_1] (\mathcal{D}^{-1} a_2 - a_2 \mathcal{D}^{-1}) = - \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\ &= -\phi_2(a_0, a_1, a_2) \end{aligned}$$

where we have used the trace property of the noncommutative integral.

(ii)  $b\phi_2(a_0, a_1, a_2, a_3)$

$$\begin{aligned}
&= \oint a_0 a_1 [\mathcal{D}, a_2] \mathcal{D}^{-1} [\mathcal{D}, a_3] \mathcal{D}^{-1} - \oint a_0 (a_1 [\mathcal{D}, a_2] + [\mathcal{D}, a_1] a_2) \mathcal{D}^{-1} [\mathcal{D}, a_3] \mathcal{D}^{-1} \\
&\quad + \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} (a_2 [\mathcal{D}, a_3] + [\mathcal{D}, a_2] a_3) \mathcal{D}^{-1} - \oint a_3 a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \oint a_0 [\mathcal{D}, a_1] (\mathcal{D}^{-1} a_2 - a_2 \mathcal{D}^{-1}) [\mathcal{D}, a_3] \mathcal{D}^{-1} + \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] (a_3 \mathcal{D}^{-1} - \mathcal{D}^{-1} a_3) \\
&= - \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} [\mathcal{D}, a_3] \mathcal{D}^{-1} + \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} [\mathcal{D}, a_3] \mathcal{D}^{-1} \\
&= 0.
\end{aligned}$$

(iii) Using Remark 2.1, we get  $\phi_3(a_0, a_1, a_2, a_3) = \oint a_0 [\mathcal{D}, a_1] [\mathcal{D}, a_2] [\mathcal{D}, a_3] |\mathcal{D}|^{-3}$ , so similar computations as for  $\phi_2$  gives  $b\phi_3 = 0$ .

(iv)  $B_0\phi_1(a_0) = \oint [\mathcal{D}, a_0] \mathcal{D}^{-1} = \oint (\mathcal{D} a_0 \mathcal{D}^{-1} - a_0) = 0$ .

$$\begin{aligned}
(v) \quad B_0\phi_2(a_0, a_1) &= \oint [\mathcal{D}, a_0] \mathcal{D}^{-1} [\mathcal{D}, a_1] \mathcal{D}^{-1} = \oint a_0 \mathcal{D}^{-1} [\mathcal{D}, a_1] - \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} \\
&= \oint a_0 a_1 - \oint a_0 \mathcal{D}^{-1} a_1 \mathcal{D} - \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} \\
&= - \oint a_1 [\mathcal{D}, a_0] \mathcal{D}^{-1} - \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} = -\phi_1(a_1, a_0) - \phi_1(a_0, a_1).
\end{aligned}$$

(vi) Since  $-b\lambda\phi_1(a_0, a_1, a_2) = \phi_1(a_2, a_0 a_1) - \phi_1(a_1 a_2, a_0) + \phi_1(a_1, a_2 a_0)$ , one obtains that

$$-b\lambda\phi_1(a_0, a_1, a_2) = \oint a_0 a_1 \mathcal{D}^{-1} a_2 \mathcal{D} + a_0 \mathcal{D}^{-1} a_1 \mathcal{D} a_2 - a_0 \mathcal{D}^{-1} a_1 a_2 \mathcal{D} - a_0 a_1 a_2.$$

So by direct expansion, this is equal to  $-\oint a_0 \mathcal{D}^{-1} [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2]$  which means that

$$\begin{aligned}
-b\lambda\phi_1(a_0, a_1, a_2) &= \oint [\mathcal{D}^{-1}, a_0] [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] - a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= -B_0\phi_3(a_0, a_1, a_2) - \phi_2(a_0, a_1, a_2).
\end{aligned}$$

Now the result follows from (i), (v).

(vii)  $B\phi_2 = NB_0\phi_2 = -N(1 - \lambda)\phi_1 = 0$  since  $N(1 - \lambda) = 0$ .

$$\begin{aligned}
(viii) \quad B_0\phi_3(a_0, a_1, a_2) &= \oint [\mathcal{D}, a_0] \mathcal{D}^{-1} [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \oint a_0 \mathcal{D}^{-1} [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] - \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \oint a_0 a_1 \mathcal{D}^{-1} [\mathcal{D}, a_2] - \oint a_0 \mathcal{D}^{-1} a_1 [\mathcal{D}, a_2] - \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \oint a_0 a_1 a_2 - \oint a_0 a_1 \mathcal{D}^{-1} a_2 \mathcal{D} - \oint a_0 \mathcal{D}^{-1} a_1 \mathcal{D} a_2 + \oint a_0 \mathcal{D}^{-1} a_1 a_2 \mathcal{D} \\
&\quad - \oint a_0 [\mathcal{D}, a_1] \mathcal{D}^{-1} [\mathcal{D}, a_2] \mathcal{D}^{-1} \\
&= \oint a_0 a_1 a_2 - a_2 \mathcal{D} a_1 a_0 \mathcal{D}^{-1} + a_1 a_2 \mathcal{D} a_0 \mathcal{D}^{-1} + a_2 \mathcal{D} a_0 a_1 \mathcal{D}^{-1} \\
&\quad - (a_0 \mathcal{D} a_1 a_2 \mathcal{D}^{-1} - a_0 \mathcal{D} a_1 \mathcal{D}^{-1} - a_0 a_1 \mathcal{D} a_2 \mathcal{D}^{-1} + a_0 a_1 a_2).
\end{aligned}$$

Expanding  $(id + \lambda + \lambda^2)b'\phi_1(a_0, a_1, a_2)$ , we recover previous expression.  
 (ix) Consequence of (viii). □

## 2.2 Scale-invariant term of the spectral action

We know from [8] that the scale-invariant term of the action can be written as

$$\zeta_{D_A}(0) - \zeta_D(0) = - \oint AD^{-1} + \frac{1}{2} \oint AD^{-1}AD^{-1} - \frac{1}{3} \oint AD^{-1}AD^{-1}AD^{-1}. \quad (5)$$

In fact, this action can be expressed in dimension 3 as contributions corresponding to tadpole and the Yang–Mills and Chern–Simons actions in dimension 4:

**Proposition 2.3.** *For any one-form  $A$ ,*

$$\zeta_{D_A}(0) - \zeta_D(0) = -\frac{1}{2} \int_{N\phi_1} A + \frac{1}{2} \int_{\phi_2} (dA + A^2) - \frac{1}{2} \int_{\phi_3} (AdA + \frac{2}{3}A^3). \quad (6)$$

To prove this, we calculate now each terms of the action.

**Lemma 2.4.** *For any one-form  $A$ , we have*

- (i)  $\int_{\phi_2} dA = \int_{B_0\phi_2} A = - \int_{\phi_1} A - \int_{\lambda\phi_1} A$ .
- (ii)  $\oint AD^{-1} = \int_{\phi_1} A = \frac{1}{2} \int_{N\phi_1} A - \frac{1}{2} \int_{\phi_2} dA$ .
- (iii)  $\oint AD^{-1}AD^{-1} = - \int_{\phi_3} AdA + \int_{\phi_2} A^2$ .
- (iv)  $\oint AD^{-1}AD^{-1}AD^{-1} = \int_{\phi_3} A^3$ .

*Proof.* (i) and (ii) follow directly from Lemma 2.2 (v).

(iii) With the shorthand  $A = a_i db_i$  (summation on  $i$ )

$$\begin{aligned} \oint AD^{-1}AD^{-1} &= \oint a_0[\mathcal{D}, b_0]\mathcal{D}^{-1}a_1[\mathcal{D}, b_1]\mathcal{D}^{-1} \\ &= - \int_{\phi_3} AdA + \oint a_0[\mathcal{D}, b_0]a_1b_1\mathcal{D}^{-1} - \oint a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}b_1. \end{aligned}$$

We calculate further the remaining terms

$$\begin{aligned} \oint a_0[\mathcal{D}, b_0]a_1b_1\mathcal{D}^{-1} - \oint a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}b_1 &= \oint a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1} - \oint a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1} \\ &\quad - \oint a_0\mathcal{D}b_0a_1\mathcal{D}^{-1}b_1 + \oint a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}b_1, \end{aligned}$$

which are compared with  $\int_{\phi_2} A^2 = \int_{\phi_2} a_0(db_0)a_1db_1 = \int_{\phi_2} a_0d(b_0a_1)db_1 - a_0b_0da_1db_1$ :

$$\begin{aligned} \int_{\phi_2} A^2 &= \oint a_0[\mathcal{D}, b_0a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1]\mathcal{D}^{-1} - \oint a_0b_0[\mathcal{D}, a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1]\mathcal{D}^{-1} \\ &= \oint a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1} - \oint a_0\mathcal{D}b_0a_1\mathcal{D}^{-1}b_1 - \oint a_0b_0a_1\mathcal{D}b_1\mathcal{D}^{-1} + \oint a_0b_0a_1b_1 \\ &\quad - \oint a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1} + \oint a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}b_1 + \oint a_0b_0a_1\mathcal{D}b_1\mathcal{D}^{-1} - \oint a_0b_0a_1b_1 \\ &= \oint a_0\mathcal{D}b_0a_1b_1\mathcal{D}^{-1} - \oint b_1a_0\mathcal{D}b_0a_1\mathcal{D}^{-1} - \oint a_0b_0\mathcal{D}a_1b_1\mathcal{D}^{-1} + \oint b_1a_0b_0\mathcal{D}a_1\mathcal{D}^{-1}. \end{aligned}$$

(iv) Note that

$$\begin{aligned}
\int_{\phi_3} A^3 &= \int_{\phi_3} a_0(db_0)a_1(db_1)a_2db_2 = \int_{\phi_3} a_0d(b_0a_1)d(b_1a_2)db_2 - a_0b_0da_1d(b_1a_2)db_2 \\
&\quad - a_0d(b_0a_1b_1)d(a_2db_2 + a_0b_0d(a_1b_1)da_2db_2 \\
&= \int a_0[\mathcal{D}, b_0a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} - a_0b_0[\mathcal{D}, a_1]\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} \\
&\quad - a_0[\mathcal{D}, b_0a_1b_1]\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} + a_0b_0[\mathcal{D}, a_1b_1]\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}.
\end{aligned}$$

Summing up the first two terms and the last two ones gives

$$\int_{\phi_3} A^3 = \int a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}[\mathcal{D}, b_1a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1} - a_0[\mathcal{D}, b_0]a_1b_1\mathcal{D}^{-1}[\mathcal{D}, a_2]\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}.$$

Using Remark 2.1, we can commute under the integral  $\mathcal{D}^{-1}$  with all  $a \in \mathcal{A}$  and similarly

$$\int A\mathcal{D}^{-1}A\mathcal{D}^{-1}A\mathcal{D}^{-1} = \int a_0[\mathcal{D}, b_0]a_1\mathcal{D}^{-1}[\mathcal{D}, b_1]a_2\mathcal{D}^{-1}[\mathcal{D}, b_2]\mathcal{D}^{-1}$$

which proves (iv).  $\square$

We deduce Proposition 2.3 from (5) using the previous lemma.

### 3 The $SU_q(2)$ triple

#### 3.1 The spectral triple

We briefly recall the main facts of the real spectral triple  $(\mathcal{A}(SU_q(2)), \mathcal{H}, \mathcal{D})$  introduced in [21], see also [4, 5, 13].

*The algebra:*

Let  $\mathcal{A} := \mathcal{A}(SU_q(2))$  be the  $*$ -algebra generated polynomially by  $a$  and  $b$ , subject to the following commutation rules with  $0 < q < 1$ :

$$ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b, \quad a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1. \quad (7)$$

**Lemma 3.1.** *For any representation  $\pi$  of  $\mathcal{A}$ ,*

$$\begin{aligned}
\text{Spect}(\pi(bb^*)) &= \{0, q^{2k} \mid k \in \mathbb{N}\} \text{ or } \pi(b) = 0, \\
\text{Spect}(\pi(aa^*)) &= \{1, 1 - q^{2k} \mid k \in \mathbb{N}\} \text{ or } \pi(b) = 0 \text{ and } \pi(a) \text{ is a unitary.}
\end{aligned}$$

*Proof.* [31] Since  $\{0\} \cup \sigma(\pi(aa^*)) = \{0\} \cup \sigma(\pi(a^*a))$ , we get

$$\{1\} \cup B = \{1\} \cup q^2B \quad (8)$$

if  $B := \sigma(\pi(bb^*))$ . Since  $0 \leq \pi(bb^*) \leq 1$ , so  $B$  is a closed subset of  $[0, 1]$ . Assume  $b \neq 0$ .

Let  $s := \sup(B)$  and suppose  $s \neq 1$ . Then  $s = q^2x$  where  $x \in B$ . Thus  $s = q^2x < x \leq s$  gives  $s = 0$  and the contradiction  $b = 0$ , thus  $1 \in B$ . Similar argument for  $\inf(B)$  implies  $0 \in B$ .

Let  $C := \{0, q^{2k} \mid k \in \mathbb{N}\} \subset B$  and assume  $B \setminus C \neq \emptyset$ . Then  $B \setminus C = (q^2B) \setminus C$  by (8) and this is equal to  $q^2(B \setminus C)$  since  $q^{-2} > 1$ . If  $s := \sup(B \setminus C)$ , then  $s = \lim_n(q^2x_n)$  where  $x_n \in B \setminus C$  and  $s = q^2 \lim_n x_n \leq q^2s$  implying  $s = 0$ . But  $B \setminus C \subset \{0\}$  yields a contradiction, so  $B \setminus C = \emptyset$ .  $\square$

This lemma is interesting since it shows the appearance of discreteness for  $0 \leq q < 1$  while for  $q = 1$ ,  $SU_q(2) = SU(2) \simeq \mathbb{S}^3$  and the spectrum of the commuting operator  $\pi(aa^*)$  and  $\pi(bb^*)$  are equal to  $[0, 1]$ . Moreover, all foregoing results on noncommutative integrals will involve  $q^2$  and not  $q$ .

Any element of  $\mathcal{A}$  can be uniquely decomposed as a linear combination of terms of the form  $a^\alpha b^\beta b^{*\gamma}$  where  $\alpha \in \mathbb{Z}$ ,  $\beta, \gamma \in \mathbb{N}$ , with the convention

$$a^{-|\alpha|} := a^{*|\alpha|}.$$

*The spinorial Hilbert space:*

$\mathcal{H} = \mathcal{H}^\uparrow \oplus \mathcal{H}^\downarrow$  has an orthonormal basis consisting of vectors  $|j\mu n\rangle^\uparrow$  with  $j = 0, \frac{1}{2}, 1, \dots$ ,  $\mu = -j, \dots, j$  and  $n = -j^+, \dots, j^+$ , together with  $|j\mu n\rangle^\downarrow$  for  $j = \frac{1}{2}, 1, \dots$ ,  $\mu = -j, \dots, j$  and  $n = -j^-, \dots, j^-$  (here  $x^\pm := x \pm \frac{1}{2}$ ).

It is convenient to use a vector notation, setting:

$$|j\mu n\rangle\rangle := \begin{pmatrix} |j\mu n\rangle^\uparrow \\ |j\mu n\rangle^\downarrow \end{pmatrix} \quad (9)$$

and with the convention that the lower component is zero when  $n = \pm(j + \frac{1}{2})$  or  $j = 0$ .

*The representation  $\pi$  and its approximate  $\underline{\pi}$ :*

It is known that representation theory of  $SU_q(2)$  is similar to that of  $SU(2)$  [39]. The representation  $\pi$  given in [21] is:

$$\begin{aligned} \pi(a) |j\mu n\rangle\rangle &:= \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle\rangle, \\ \pi(b) |j\mu n\rangle\rangle &:= \beta_{j\mu n}^+ |j^+ \mu^+ n^-\rangle\rangle + \beta_{j\mu n}^- |j^- \mu^+ n^-\rangle\rangle, \\ \pi(a^*) |j\mu n\rangle\rangle &:= \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^- n^-\rangle\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^- n^-\rangle\rangle, \\ \pi(b^*) |j\mu n\rangle\rangle &:= \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^+\rangle\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^+\rangle\rangle \end{aligned} \quad (10)$$

where

$$\begin{aligned} \alpha_{j\mu n}^+ &:= \sqrt{q^{\mu+n-1/2} [j + \mu + 1]} \begin{pmatrix} q^{-j-1/2} \frac{\sqrt{[j+n+3/2]}}{[2j+2]} & 0 \\ q^{1/2} \frac{\sqrt{[j-n+1/2]}}{[2j+1][2j+2]} & q^{-j} \frac{\sqrt{[j+n+1/2]}}{[2j+1]} \end{pmatrix}, \\ \alpha_{j\mu n}^- &:= \sqrt{q^{\mu+n+1/2} [j - \mu]} \begin{pmatrix} q^{j+1} \frac{\sqrt{[j-n+1/2]}}{[2j+1]} & -q^{1/2} \frac{\sqrt{[j+n+1/2]}}{[2j][2j+1]} \\ 0 & q^{j+1/2} \frac{\sqrt{[j-n-1/2]}}{[2j]} \end{pmatrix}, \\ \beta_{j\mu n}^+ &:= \sqrt{q^{\mu+n-1/2} [j + \mu + 1]} \begin{pmatrix} \frac{\sqrt{[j-n+3/2]}}{[2j+2]} & 0 \\ -q^{-j-1} \frac{\sqrt{[j+n+1/2]}}{[2j+1][2j+2]} & q^{-1/2} \frac{\sqrt{[j-n+1/2]}}{[2j+1]} \end{pmatrix}, \\ \beta_{j\mu n}^- &:= \sqrt{q^{\mu+n-1/2} [j - \mu]} \begin{pmatrix} -q^{-1/2} \frac{\sqrt{[j+n+1/2]}}{[2j+1]} & -q^j \frac{\sqrt{[j-n+1/2]}}{[2j][2j+1]} \\ 0 & -\frac{\sqrt{[j+n-1/2]}}{[2j]} \end{pmatrix} \end{aligned}$$

with  $\tilde{\alpha}_{j\mu n}^\pm := (\alpha_{j^\pm \mu^\pm n^\pm}^\mp)^*$ ,  $\tilde{\beta}_{j\mu n}^\pm := (\beta_{j^\pm \mu^\pm n^\pm}^\mp)^*$  and with the  $q$ -number of  $\alpha \in \mathbb{R}$  be defined as

$$[\alpha] := \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}}.$$



For the purpose of this paper it is sufficient to use the approximate spinorial \*-representation  $\underline{\pi}$  of  $SU_q(2)$  presented in [21, 38] instead of the full spinorial one  $\pi$ .

This approximate representation is

$$\underline{\pi}(a) := a_+ + a_-, \quad \underline{\pi}(b) := b_+ + b_-$$

with the following definitions:

$$\begin{aligned} a_+ |j\mu n\rangle\rangle &:= q_{j++\mu+} \begin{pmatrix} q_{j++n+1} & 0 \\ 0 & q_{j++n} \end{pmatrix} |j^+ \mu^+ n^+\rangle\rangle, \\ a_- |j\mu n\rangle\rangle &:= q^{2j+\mu+n+\frac{1}{2}} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^- \mu^+ n^+\rangle\rangle, \\ b_+ |j\mu n\rangle\rangle &:= q^{j+n-\frac{1}{2}} q_{j++\mu+} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} |j^+ \mu^+ n^-\rangle\rangle, \\ b_- |j\mu n\rangle\rangle &:= -q^{j+\mu} \begin{pmatrix} q_{j++n} & 0 \\ 0 & q_{j--n} \end{pmatrix} |j^- \mu^+ n^-\rangle\rangle. \end{aligned} \quad (11)$$

All disregarded terms are trace-class and do not influence residue calculations. More precisely,  $\pi(x) - \underline{\pi}(x) \in \mathcal{K}_q$  where  $\mathcal{K}_q$  is the principal ideal generated by the operators

$$J_q |j\mu n\rangle\rangle := q^j |j\mu n\rangle\rangle. \quad (12)$$

Actually,  $\mathcal{K}_q$  is independent of  $q$  and is contained in all ideals of operators such that  $\mu_n = o(n^{-\alpha})$  (infinitesimal of order  $\alpha$ ) for any  $\alpha > 0$ , and  $\mathcal{K}_q \subset OP^{-\infty}$ .

We define the alternative orthonormal basis  $v_{m,l}^{j\uparrow}$  and  $v_{m,l}^{j\downarrow}$  and the vector notation

$$v_{m,l}^j := \begin{pmatrix} v_{m,l}^{j\uparrow} \\ v_{m,l}^{j\downarrow} \end{pmatrix} \text{ where } v_{m,l}^{j\uparrow} := |j, m-j, l-j^+, \uparrow\rangle, \quad v_{m,l}^{j\downarrow} := |j, m-j, l-j^-, \downarrow\rangle.$$

Here  $j \in \frac{1}{2}\mathbb{N}$ ,  $0 \leq m \leq 2j$ ,  $0 \leq l \leq 2j+1$  and  $v_{m,l}^{j\downarrow}$  is zero whenever  $j = 0$  or  $l = 2j$  or  $2j+1$ . The interest is that now, the operators  $a_{\pm}$  and  $b_{\pm}$  satisfy the simpler relations

$$\begin{aligned} a_+ v_{m,l}^j &= q_{m+1} q_{l+1} v_{m+1,l+1}^{j+}, & a_- v_{m,l}^j &= q^{m+l+1} v_{m,l}^{j-}, \\ b_+ v_{m,l}^j &= q^l q_{m+1} v_{m+1,l}^{j+}, & b_- v_{m,l}^j &= -q^m q_l v_{m,l-1}^{j-}. \end{aligned} \quad (13)$$

Thus

$$\begin{aligned} a_+^* v_{m,l}^j &= q_m q_l v_{m-1,l-1}^{j-}, & a_-^* v_{m,l}^j &= q^{m+l+1} v_{m,l}^{j+}, \\ b_+^* v_{m,l}^j &= q^l q_m v_{m-1,l}^{j-}, & b_-^* v_{m,l}^j &= -q^m q_{l+1} v_{m,l+1}^{j+}. \end{aligned} \quad (14)$$

Moreover, we have

$$\begin{aligned} a_- a_+ &= q^2 a_+ a_-, & b_- b_+ &= q^2 b_+ b_-, & b_+ a_+ &= q a_+ b_+, & b_- a_- &= q a_- b_-, \\ a_-^* a_+ &= q^2 a_+ a_-^*, & a_-^* a_- &= a_- a_-^*, & a_-^* b_+ &= q b_+ a_-^*, & a_-^* b_- &= q b_- a_-^*, \\ a_+^* a_- &= q^2 a_- a_+^*, & b_-^* b_+ &= b_+ b_-^*, & b_-^* a_+ &= q a_+ b_-^*, & a_- b_+ &= q b_+ a_- . \end{aligned} \quad (15)$$

Note for instance that

$$\begin{aligned} a_+ a_+^* v_{m,l}^j &= q_m^2 q_l^2 v_{m,l}^j, & a_+^* a_+ v_{m,l}^j &= q_{m+1}^2 q_{l+1}^2 v_{m,l}^j, \\ b_+ b_+^* v_{m,l}^j &= q^{2l} q_m^2 v_{m,l}^j, & b_+^* b_+ v_{m,l}^j &= q^{2l} q_{m+1}^2 v_{m,l}^j, \end{aligned}$$

so applied to  $v_{m,l}^j$ , we get the first relation (and similarly for the others)

$$a_+^* a_+ - q^2 a_+ a_+^* + q^2 (b_+^* b_+ - b_+ b_+^*) = 1 - q^2, \quad (16)$$

$$a_+ a_+^* + a_- a_-^* + b_+ b_+^* + b_- b_-^* = 1, \quad (17)$$

$$a_+^* a_+ + a_-^* a_- + q^2 (b_+^* b_+ + b_-^* b_-) = 1, \quad (18)$$

$$a_-^* a_- - q^2 a_- a_-^* + q^2 b_-^* b_- - q^2 b_- b_-^* = 0, \quad (19)$$

$$a_+ a_-^* + b_-^* b_+ = 0, \quad a_-^* a_+ + q^2 b_-^* b_+ = 0, \quad (20)$$

$$a_- a_+^* + b_+^* b_- = 0, \quad a_+^* a_- + q^2 b_+^* b_- = 0, \quad (21)$$

$$b_+ b_+^* - b_+^* b_+ + b_- b_-^* - b_-^* b_- = 0, \quad (22)$$

$$q a_+ b_- - b_- a_+ + q a_- b_+ - b_+ a_- = 0. \quad (23)$$

*And two others:*

Note that we also use two other infinite dimensional  $*$ -representations  $\pi_\pm$  of  $\mathcal{A}$  on  $\ell^2(\mathbb{N})$  defined as follows on the orthonormal basis  $\{\varepsilon_n : n \in \mathbb{N}\}$  of  $\ell^2(\mathbb{N})$  by

$$\begin{aligned} \pi_\pm(a) \varepsilon_n &:= q_{n+1} \varepsilon_{n+1}, & \pi_\pm(b) \varepsilon_n &:= \pm q^n \varepsilon_n, \\ q_n &:= \sqrt{1 - q^{2n}}. \end{aligned} \quad (24)$$

These representations are irreducible but not faithful since for instance  $\pi_\pm(b - b^*) = 0$ .

*The Dirac operator:*

It is chosen the same as in the classical case of a 3-sphere with the round metric:

$$\mathcal{D} |j\mu n\rangle := \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} |j\mu n\rangle, \quad (25)$$

which means, with our convention, that  $\mathcal{D} v_{ml}^j = \begin{pmatrix} 2j+\frac{3}{2} & 0 \\ 0 & -2j-\frac{1}{2} \end{pmatrix} v_{ml}^j$ . Note that this operator is asymptotically diagonal with linear spectrum and

the eigenvalues  $2j + \frac{1}{2}$  for  $j \in \frac{1}{2}\mathbb{N}$ , have multiplicities  $(2j+1)(2j+2)$ ,

the eigenvalues  $-(2j + \frac{1}{2})$  for  $j \in \frac{1}{2}\mathbb{N}^*$ , have multiplicities  $2j(2j+1)$ .

So this Dirac operator coincide exactly with the classical one on the 3-sphere (see [1, 32]) with a gap around 0.

Let  $\mathcal{D} = F|\mathcal{D}|$  be the polar decomposition of  $\mathcal{D}$ , thus

$$|\mathcal{D}| |j\mu n\rangle = \begin{pmatrix} d_{j+} & 0 \\ 0 & d_j \end{pmatrix} |j\mu n\rangle, \quad d_j := 2j + \frac{1}{2}, \quad (26)$$

$$F |j\mu n\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} |j\mu n\rangle, \quad (27)$$

and it follows from (11) and (27) that

$$F \text{ commutes with } a_\pm, b_\pm. \quad (28)$$

*The reality operator:*

This antilinear operator  $J$  is defined on the basis of  $\mathcal{H}$  by

$$J |j, \mu, n, \uparrow\rangle := i^{2(2j+\mu+n)} |j, -\mu, -n, \uparrow\rangle, \quad J |j, \mu, n, \downarrow\rangle := i^{2(2j-\mu-n)} |j, -\mu, -n, \downarrow\rangle \quad (29)$$

thus it satisfies

$$\begin{aligned} J^{-1} &= -J = J^* \text{ and } \mathcal{D}J = J\mathcal{D}, \\ Jv_{m,l}^{j\uparrow} &= i^{2(m+l)-1}v_{2j-m,2j+1-l}^{j\uparrow}, \quad Jv_{m,l}^{j\downarrow} = i^{-2(m+l)+1}v_{2j-m,2j-1-l}^{j\downarrow}. \end{aligned}$$

*The Hopf map  $r$*

For the explicit calculations of residues, we need a  $*$ -homomorphism  $r : X \rightarrow \pi_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})$  defined by the tensor product in the sense of Hopf algebras of representations  $\pi_+$  and  $\pi_-$ :

$$\begin{aligned} r(a_+) &:= \pi_+(a) \otimes \pi_-(a), & r(a_-) &:= -q \pi_+(b) \otimes \pi_-(b^*), \\ r(b_+) &:= -\pi_+(a) \otimes \pi_-(b), & r(b_-) &:= -\pi_+(b) \otimes \pi_-(a^*). \end{aligned} \quad (30)$$

In fact,  $\mathcal{A}$  is a Hopf  $*$ -algebra under the coproduct  $\Delta(a) := a \otimes a - q b \otimes b^*$ ,  $\Delta(b) := a \otimes b + b \otimes a^*$ . These homomorphisms appeared in [39] with the translation  $\alpha \leftrightarrow a^*$ ,  $\gamma \leftrightarrow -b$ . In particular, if  $U := \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$  is the canonical generator of the  $K_1(\mathcal{A})$ -group  $(\Delta a, \Delta b) = (a, b) \dot{\otimes} U$  where the last  $\dot{\otimes}$  means the matrix product of tensors of components.

*The grading:*

According to the shift  $j \rightarrow j^\pm$  appearing in formulae (13), (14), we get a  $\mathbb{Z}$ -grading on  $X$  defined by the degree  $+1$  on  $a_+, b_+, a_-^*, b_-^*$  and  $-1$  on  $a_-, b_-, a_+^*, b_+^*$ .

Any operator  $T \in X$  can be (uniquely) decomposed as  $T = \sum_{j \in \mathbb{Z}} T_j$  where  $T_j$  is homogeneous of degree  $j$ .

For  $T \in X$ ,  $T^\circ$  will denote the 0-degree part of  $T$  for this grading and by a slight abuse of notations, we write  $r(T)^\circ$  instead of  $r(T^\circ)$ .

*The symbol map:*

We also use the  $*$ -homomorphism  $\sigma : \pi_\pm(\mathcal{A}) \rightarrow C^\infty(S^1)$  defined for  $z \in S^1$  on the generators by

$$\sigma(\pi_\pm(a))(z) := z, \quad \sigma(\pi_\pm(a^*))(z) := \bar{z}, \quad \sigma(\pi_\pm(b))(z) = \sigma(\pi_\pm(b^*))(z) := 0.$$

The application  $(\sigma \otimes \sigma) \circ r$  is defined on  $X$  (and so on  $\mathcal{B}$ ) with values in  $C^\infty(S^1) \otimes C^\infty(S^1)$ .

We define

$$dT := [\mathcal{D}, T] \text{ and } \delta(T) := [[\mathcal{D}], T].$$

**Lemma 3.2.**  $a_\pm, b_\pm$  are bounded operators on  $\mathcal{H}$  such that for all  $p \in \mathbb{N}$ ,

- (i)  $\delta(a_\pm) = \pm a_\pm$ ,  $\delta(b_\pm) = \pm b_\pm$ ,
- (ii)  $\delta^p(\pi(a)) = a_+ + (-1)^p a_-$ ,  $\delta^p(\pi(b)) = b_+ + (-1)^p b_-$ ,
- (iii)  $\delta(a_\pm^p) = \pm p a_\pm^{p-1}$ ,  $\delta(b_\pm^p) = \pm p b_\pm^{p-1}$ .

*Proof.* (i) By definition,  $a_\pm |j\mu n\rangle = \begin{pmatrix} \alpha_\pm & 0 \\ 0 & \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle$  where the numbers  $\alpha_\pm$  and  $\beta_\pm$  depend on  $j, \mu, n$  and  $q$ , so we get by (26)

$$\begin{aligned} \delta(a_\pm) |j\mu n\rangle &= \begin{pmatrix} (d_{j^\pm})^{\alpha_\pm} & 0 \\ 0 & d_{j^\pm}^{\beta_\pm} \end{pmatrix} |j^\pm \mu^+ n^+\rangle - \begin{pmatrix} (d_{j^\pm})^{\alpha_\pm} & 0 \\ 0 & d_{j^\pm}^{\beta_\pm} \end{pmatrix} |j^\pm \mu^+ n^+\rangle \\ &= \begin{pmatrix} \pm \alpha_\pm & 0 \\ 0 & \pm \beta_\pm \end{pmatrix} |j^\pm \mu^+ n^+\rangle = \pm a_\pm |j\mu n\rangle \end{aligned}$$

and similar proofs for  $b_\pm$ .

(ii) and (iii) are straightforward consequences of (i) and definition of  $\pi$ . □

We note

- $\mathcal{B}$  the  $*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$  generated by the operators in  $\delta^k(\pi(\mathcal{A}))$  for all  $k \in \mathbb{N}$ ,
- $\Psi_0^0(\mathcal{A})$  the algebra generated by  $\delta^k(\pi(\mathcal{A}))$  and  $\delta^k([\mathcal{D}, \pi(\mathcal{A})])$  for all  $k \in \mathbb{N}$ ,
- $X$  the  $*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$  algebraically generated by the set  $\{a_{\pm}, b_{\pm}\}$ .

**Remark 3.3.** By Lemma 3.2, we see that, modulo  $OP^{-\infty}$ ,  $X$  is equal to  $\mathcal{B}$  and in particular contains  $\pi(\mathcal{A})$ .

Using (28), we get that  $\mathcal{B} \subset \Psi_0^0(\mathcal{A}) \subset$  algebra generated by  $\mathcal{B}$  and  $\mathcal{B}F$ .

Note that, despite the last inclusion,  $F$  is not a priori in  $\Psi_0^0(\mathcal{A})$ .

### 3.2 The noncommutative integrals

Recall that for any pseudodifferential operator  $T$ ,  $\oint T := \text{Res}_{s=0} \zeta_D^T(s)$  where  $\zeta_D^T(s) := \text{Tr}(T|\mathcal{D}|^{-s})$ .

**Theorem 3.4.** The dimension spectrum (without reality structure given by  $J$ ) of the spectral triple  $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$  is simple and equal to  $\{1, 2, 3\}$ .

Moreover, the corresponding residues for  $T \in \mathcal{B}$  are

$$\begin{aligned} \oint T|\mathcal{D}|^{-3} &= 2(\tau_1 \otimes \tau_1)(r(T)^\circ), \\ \oint T|\mathcal{D}|^{-2} &= 2(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(T)^\circ), \\ \oint T|\mathcal{D}|^{-1} &= (2\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1)(r(T)^\circ), \\ \oint FT|\mathcal{D}|^{-3} &= 0, \\ \oint FT|\mathcal{D}|^{-2} &= 0, \\ \oint FT|\mathcal{D}|^{-1} &= (\tau_0 \otimes \tau_1 - \tau_1 \otimes \tau_0)(r(T)^\circ), \end{aligned}$$

where the functionals  $\tau_0, \tau_1$  are defined for  $x \in \pi_{\pm}(\mathcal{A})$  by

$$\tau_0(x) := \lim_{N \rightarrow \infty} (\text{Tr}_N x - (N+1)\tau_1(x)), \quad \tau_1(x) := \frac{1}{2\pi} \int_0^{2\pi} \sigma(x)(e^{i\theta}) d\theta,$$

with  $\text{Tr}_N x = \sum_{n=0}^N \langle \varepsilon_n, x \varepsilon_n \rangle$ .

*Proof.* Consequence of [38, Theorem 4.1 and (4.3)]. □

**Remark 3.5.** Since  $F$  is not in  $\mathcal{B}$ , the equation of Theorem 3.4 are not valid for all  $T \in \Psi_0^0(\mathcal{A})$ . But when  $T \in \Psi_0^0(\mathcal{A})$ ,  $\oint T|\mathcal{D}|^{-k} = 0$  for  $k \notin \{1, 2, 3\}$  since the dimension spectrum is  $\{1, 2, 3\}$  [38].

Compared to [38] where we had

$$\tau_0^\uparrow(x) := \lim_{N \rightarrow \infty} \text{Tr}_N x - (N + \frac{3}{2})\tau_1(x), \quad \tau_0^\downarrow(x) := \lim_{N \rightarrow \infty} \text{Tr}_N x - (N + \frac{1}{2})\tau_1(x),$$

we replaced them with  $\tau_0$ :

$$\tau_0^\uparrow = \tau_0 - \frac{1}{2}\tau_1, \quad \tau_0^\downarrow = \tau_0 + \frac{1}{2}\tau_1.$$

Note that  $\tau_1$  is a trace on  $\pi_\pm(\mathcal{A})$  such that  $\tau_1(1) = 1$ , while  $\tau_0$  is not since  $\tau_0(1) = 0$  and

$$\tau_0(\pi_\pm(aa^*)) = \lim_{N \rightarrow \infty} \sum_{n=0}^{\infty} (1 - q^{2n}) - (N + 1) = -\frac{1}{1-q^2}, \quad (31)$$

so, because of the shift, the replacement  $a \leftrightarrow a^*$  gives

$$\tau_0(\pi_\pm(a^*a)) = q^2 \tau_0(\pi_\pm(aa^*)). \quad (32)$$

### 3.3 The tadpole

**Lemma 3.6.** *For  $SU_q(2)$ , the condition of the vanishing tadpole (see [15]) is not satisfied.*

*Proof.* For example, an explicit calculation gives  $\oint \pi(b)[\mathcal{D}, \pi(b^*)]\mathcal{D}^{-1} = \frac{2}{1-q^2}$ :

Let  $x, y \in \underline{\pi}(\mathcal{A})$ . Since  $[F, x] = 0$ , we have

$$\oint x[\mathcal{D}, y]\mathcal{D}^{-1} = \oint x\delta(y)|\mathcal{D}|^{-1} = \tau'(r(x\delta(y))^0)$$

where  $\tau' := 2\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1$ .

By Lemma 3.2,  $\underline{\pi}(b)\delta(\underline{\pi}(b^*)) = (b_+ + b_-)((b_-)^* - (b_+)^*) = -b_+b_+^* + b_-b_-^* + b_+b_-^* - b_-b_+^*$ . Since only the first two terms have degree 0, we get, using the formulae from Theorem 3.4

$$\begin{aligned} \tau'(r(-b_+b_+^*)) &= -\tau'(\pi_+(aa^*) \otimes \pi_-(bb^*)) \\ &= -2\tau_0(\pi_+(aa^*))\tau_0(\pi_-(bb^*)) + \frac{1}{2}\tau_1(\pi_+(aa^*))\tau_1(\pi_-(bb^*)) \end{aligned}$$

and  $\tau_1(\pi_+(aa^*)) = \frac{1}{2\pi} \int_0^{2\pi} 1 d\theta = 1$ ,  $\tau_1(\pi_-(bb^*)) = 0$ . Similarly, using (32)

$$\tau'(r(-b_-b_-^*)) = 2\tau_0(\pi_+(bb^*))\tau_0(\pi_-(a^*a)) = 2q^2\tau_0(\pi_-(aa^*))\tau_0(\pi_+(bb^*)).$$

Since  $\tau_0(\pi_\pm(bb^*)) = \text{Tr}(\pi_\pm(bb^*)) = \sum_{n=0}^{\infty} q^{2n} = \frac{1}{1-q^2}$  and (31),

$$\oint \pi(b)[\mathcal{D}, \pi(b^*)]\mathcal{D}^{-1} = 2\frac{1}{1-q^2}\frac{1}{1-q^2} + 2q^2\frac{-1}{1-q^2}\frac{1}{1-q^2} = \frac{2}{1-q^2}. \quad \square$$

In particular the pairing of the tadpole cyclic cocycle  $\phi_1$  with the generator of  $K_1$ -group is nontrivial:

**Remark 3.7.** *Other examples: with the shortcut  $x$  instead of  $\underline{\pi}(x)$ ,*

$$\begin{aligned} (\tau_1 \otimes \tau_1) r(a\delta(a^*)^\circ) &= -1, & (\tau_1 \otimes \tau_1) r(a^*\delta(a)^\circ) &= 1, \\ (\tau_0 \otimes \tau_0) r(a\delta(a^*)^\circ) &= \frac{1}{q^2-1}, & (\tau_0 \otimes \tau_0) r(a^*\delta(a)^\circ) &= \frac{q^2}{q^2-1}, \\ \oint a\delta(a^*)|\mathcal{D}|^{-1} &= \frac{q^2+3}{2(q^2-1)}, & \oint a^*\delta(a)|\mathcal{D}|^{-1} &= \frac{3q^2+1}{2(q^2-1)}, \\ \oint b\delta(b)|\mathcal{D}|^{-1} &= 0, & \oint b^*\delta(b^*)|\mathcal{D}|^{-1} &= 0, \\ \oint b\delta(b^*)|\mathcal{D}|^{-1} &= \frac{-2}{q^2-1}, & \oint b^*\delta(b)|\mathcal{D}|^{-1} &= \frac{-2}{q^2-1}. \end{aligned}$$

In particular,  $N\Phi_1$  does not vanish on 1-forms since  $\int_{N\Phi_1} ada^* = N\Phi_1(a, a^*) = -1$ .

Let  $U$  be the canonical generator of the  $K_1(\mathcal{A})$ -group,  $U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}$  acting on  $\mathcal{H} \otimes \mathbb{C}^2$ . Then for  $A_U := \sum_{k,l=1}^2 \pi(U_{kl}) d\pi(U_{kl}^*)$ , using above remark,  $\int_{\phi_1} A_U = -2$  as obtained in [38, page 391]: in fact, with  $P := \frac{1}{2}(1 + F)$ ,

$$\psi_1(U, U^*) := 2 \sum_{k,l} \oint U_{kl} \delta(U_{kl}^*) P |\mathcal{D}|^{-1} - \oint U_{kl} \delta^2(U_{kl}^*) P |\mathcal{D}|^{-2} + \frac{2}{3} \oint U_{kl} \delta^3(U_{kl}^*) P |\mathcal{D}|^{-3}$$

satisfies  $\psi_1(U, U^*) = 2 \sum_{k,l} \oint U_{kl} \delta(U_{kl}^*) P |\mathcal{D}|^{-1} = \int_{\phi_1} A_U$ .

## 4 Reality operator and spectral action on $SU_q(2)$

### 4.1 Spectral action in dimension 3 with $[F, \mathcal{A}] \in OP^{-\infty}$

Let  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  a be real spectral triple of dimension 3. Assume that  $[F, \mathcal{A}] \in OP^{-\infty}$ , where  $F := \mathcal{D}|\mathcal{D}|^{-1}$  (we suppose  $\mathcal{D}$  invertible). Let  $\mathbb{A}$  be a selfadjoint one form, so  $\mathbb{A}$  is of the form  $\mathbb{A} = \sum_i a_i db_i$  where  $a_i, b_i \in \mathcal{A}$ .

Thus,  $\mathbb{A} \simeq AF \pmod{OP^{-\infty}}$  where  $A := \sum_i a_i \delta(b_i)$  is the  $\delta$ -one-form associated to  $\mathbb{A}$ . Note that  $A$  and  $F$  commute modulo  $OP^{-\infty}$ .

We define

$$\begin{aligned} D_{\mathbb{A}} &:= \mathcal{D}_{\mathbb{A}} + P_{\mathbb{A}}, \quad P_{\mathbb{A}} \text{ the projection on } \text{Ker } \mathcal{D}_{\mathbb{A}}, \\ \mathcal{D}_{\mathbb{A}} &:= \mathcal{D} + \tilde{\mathbb{A}}, \quad \tilde{\mathbb{A}} := \mathbb{A} + J\mathbb{A}J^{-1}. \end{aligned}$$

**Theorem 4.1.** *The coefficients of the full spectral action (with reality operator) on any real spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  of dimension 3 such that  $[F, \mathcal{A}] \in OP^{-\infty}$  are*

$$\begin{aligned} (i) \quad & \oint |D_{\mathbb{A}}|^{-3} = \oint |\mathcal{D}|^{-3}. \\ (ii) \quad & \oint |D_{\mathbb{A}}|^{-2} = \oint |\mathcal{D}|^{-2} - 4 \oint A |\mathcal{D}|^{-3}. \\ (iii) \quad & \oint |D_{\mathbb{A}}|^{-1} = \oint |\mathcal{D}|^{-1} - 2 \oint A |\mathcal{D}|^{-2} + 2 \oint A^2 |\mathcal{D}|^{-3} + 2 \oint AJAJ^{-1} |\mathcal{D}|^{-3}. \\ (iv) \quad & \zeta_{D_{\mathbb{A}}}(0) = \zeta_{\mathcal{D}}(0) - 2 \oint A |\mathcal{D}|^{-1} + \oint A(A + JAJ^{-1}) |\mathcal{D}|^{-2} + \oint \delta(A)(A + JAJ^{-1}) |\mathcal{D}|^{-3} \\ & \quad - \frac{2}{3} \oint A^3 |\mathcal{D}|^{-3} - 2 \oint A^2 JAJ^{-1} |\mathcal{D}|^{-3}. \end{aligned}$$

*Proof.* (i) We apply [22, Proposition 4.9].

(ii) By [22, Lemma 4.10 (i)], we have  $\oint |D_{\mathbb{A}}|^{-2} = \oint |\mathcal{D}|^{-2} - \oint (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2) |\mathcal{D}|^{-4}$ . By the trace property of the noncommutative integral and the fact that  $\tilde{\mathbb{A}}^2 |\mathcal{D}|^{-4}$  is trace-class, we get  $\oint |D_{\mathbb{A}}|^{-2} = \oint |\mathcal{D}|^{-2} - 2 \oint \tilde{\mathbb{A}}\mathcal{D} |\mathcal{D}|^{-4} = \oint |\mathcal{D}|^{-2} - 4 \oint A |\mathcal{D}|^{-3}$ . Since  $\mathbb{A}\mathcal{D} \sim A |\mathcal{D}| \pmod{OP^{-\infty}}$ , we get the result.

(iii) By [22, Lemma 4.10 (ii)], we have

$$\oint |D_{\mathbb{A}}|^{-1} = \oint |\mathcal{D}|^{-1} - \frac{1}{2} \oint (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2) |\mathcal{D}|^{-3} + \frac{3}{8} \oint (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2)^2 |\mathcal{D}|^{-5}.$$

Following arguments of (ii), we get

$$\begin{aligned}\oint (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2)|\mathcal{D}|^{-3} &= 4\oint A|\mathcal{D}|^{-2} + 2\oint A^2|\mathcal{D}|^{-3} + 2\oint AJAJ^{-1}|\mathcal{D}|^{-3}, \\ \oint (\tilde{\mathbb{A}}\mathcal{D} + \mathcal{D}\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2)^2|\mathcal{D}|^{-5} &= 8\oint A^2|\mathcal{D}|^{-3} + 8\oint AJAJ^{-1}|\mathcal{D}|^{-3},\end{aligned}$$

and the result follows.

(iv) By [22, Lemma 4.5] gives  $\zeta_{D_{\mathbb{A}}}(0) = \sum_{j=1}^3 \frac{(-1)^j}{j} \oint (\tilde{\mathbb{A}}\mathcal{D}^{-1})^j$ .

Moreover, we have  $\oint \tilde{\mathbb{A}}\mathcal{D}^{-1} = 2\oint A|\mathcal{D}|^{-1}$  and  $\oint (\tilde{\mathbb{A}}\mathcal{D}^{-1})^2 = 2\oint (A|\mathcal{D}|^{-1})^2 + 2\oint A|\mathcal{D}|^{-1}AJAJ^{-1}|\mathcal{D}|^{-1}$ . Since  $\delta(A) \in OP^0$ , we can check that  $\oint (A|\mathcal{D}|^{-1})^2 = \oint A^2|\mathcal{D}|^{-2} + \oint \delta(A)A|\mathcal{D}|^{-3}$  and, with the same argument, that  $\oint A|\mathcal{D}|^{-1}AJAJ^{-1}|\mathcal{D}|^{-1} = \oint AJAJ^{-1}|\mathcal{D}|^{-2} + \oint \delta(A)AJAJ^{-1}|\mathcal{D}|^{-3}$ . Thus, we get

$$\oint (\tilde{\mathbb{A}}\mathcal{D}^{-1})^2 = 2\oint A(A + AJAJ^{-1})|\mathcal{D}|^{-2} + 2\oint \delta(A)(A + AJAJ^{-1})|\mathcal{D}|^{-3}. \quad (33)$$

The third term to be computed is

$$\oint (\tilde{\mathbb{A}}\mathcal{D}^{-1})^3 = 2\oint (A|\mathcal{D}|^{-1})^3 + 4\oint (A|\mathcal{D}|^{-1})^2 AJAJ^{-1}|\mathcal{D}|^{-1} + 2\oint A|\mathcal{D}|^{-1}AJAJ^{-1}|\mathcal{D}|^{-1}A|\mathcal{D}|^{-1}.$$

Any operator in  $OP^{-4}$  being trace-class here, we get

$$\oint (\tilde{\mathbb{A}}\mathcal{D}^{-1})^3 = 2\oint A^3|\mathcal{D}|^{-3} + 4\oint A^2AJAJ^{-1}|\mathcal{D}|^{-3} + 2\oint AJAJ^{-1}A|\mathcal{D}|^{-3}. \quad (34)$$

Since  $\oint AJAJ^{-1}A|\mathcal{D}|^{-3} = \oint A^2AJAJ^{-1}|\mathcal{D}|^{-3}$  by trace property and the fact that  $\delta(A) \in OP^0$ , the result follows then from (33) and (34).  $\square$

**Corollary 4.2.** *For the spectral action of  $\mathbb{A}$  without the reality operator (i.e.  $\mathcal{D}_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$ ), we get*

$$\begin{aligned}\oint |D_{\mathbb{A}}|^{-2} &= \oint |\mathcal{D}|^{-2} - 2\oint A|\mathcal{D}|^{-3}, \\ \oint |D_{\mathbb{A}}|^{-1} &= \oint |\mathcal{D}|^{-1} - \oint A|\mathcal{D}|^{-2} + \oint A^2|\mathcal{D}|^{-3}, \\ \zeta_{D_{\mathbb{A}}}(0) &= \zeta_{\mathcal{D}}(0) - \oint A|\mathcal{D}|^{-1} + \frac{1}{2}\oint A^2|\mathcal{D}|^{-2} + \frac{1}{2}\oint \delta(A)A|\mathcal{D}|^{-3} - \frac{1}{3}\oint A^3|\mathcal{D}|^{-3}.\end{aligned}$$

## 4.2 Spectral action on $SU_q(2)$ : main result

On  $SU_q(2)$ , since  $F$  commutes with  $a_{\pm}$  and  $b_{\pm}$ , the previous lemma can be used for the spectral action computation.

Here is the main result of this section

**Theorem 4.3.** *In the full spectral action (4) (with the reality operator) of  $SU_q(2)$  for a one-form*

$\mathbb{A}$  and  $A$  its associated  $\delta$ -one-form, the coefficients are:

$$\begin{aligned}
\oint |D_{\mathbb{A}}|^{-3} &= 2, \\
\oint |D_{\mathbb{A}}|^{-2} &= -4 \oint A|\mathcal{D}|^{-3}, \\
\oint |D_{\mathbb{A}}|^{-1} &= -\frac{1}{2} + 2\left(\oint A^2|\mathcal{D}|^{-3} - \oint A|\mathcal{D}|^{-2}\right) + \left|\oint A|\mathcal{D}|^{-3}\right|^2, \\
\zeta_{D_{\mathbb{A}}}(0) &= -2 \oint A|\mathcal{D}|^{-1} + \oint A^2|\mathcal{D}|^{-2} - \frac{2}{3} \oint A^3|\mathcal{D}|^{-3} \\
&\quad + \overline{\oint A|\mathcal{D}|^{-3}} \left(\frac{1}{2} \oint A|\mathcal{D}|^{-2} - \oint A^2|\mathcal{D}|^{-3}\right) + \frac{1}{2} \oint A|\mathcal{D}|^{-3} \overline{\oint A|\mathcal{D}|^{-2}}.
\end{aligned}$$

In order to prove this theorem, we will use a decomposition of one-forms in the Poincaré–Birkhoff–Witt basis of  $\mathcal{A}$  with an extension of previous representations to operators like  $TJT'J^{-1}$  where  $T$  and  $T'$  are in  $X$ .

### 4.3 Balanced components and Poincaré–Birkhoff–Witt basis of $\mathcal{A}$

Our objective is to compute all integrals in term of  $A$  and the computation will lead to functions of  $A$  which capture certain symmetries on  $\mathcal{A}$ .

For convenience, let us introduce now these functions:

Let  $\mathbb{A} = \sum_i \pi(x^i) d\pi(y^i)$  on  $SU_q(2)$  be one-form and  $A$  the associated  $\delta$ -one-form. The  $x^i$  and  $y^i$  are in  $\mathcal{A}$  and as such they can be uniquely written as finite sums  $x^i = \sum_{\alpha} x_{\alpha}^i m^{\alpha}$  and  $y^i = \sum_{\beta} y_{\beta}^i m^{\beta}$  where  $m^{\alpha} := a^{\alpha_1} b^{\alpha_2} b^{*\alpha_3}$  is the canonical monomial of  $\mathcal{A}$  with  $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$  based on a fixed Poincaré–Birkhoff–Witt type basis of  $\mathcal{A}$ .

**Remark 4.4.** Any one-form  $\mathbb{A} = \sum_i \pi(x^i) d\pi(y^i)$  on  $SU_q(2)$  is characterized by a complex valued matrix  $A_{\alpha}^{\beta} = \sum_i x_{\alpha}^i y_{\beta}^i$  where  $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ . This matrix is such that

$$A = A_{\alpha}^{\beta} M_{\beta}^{\alpha}$$

where  $M_{\beta}^{\alpha} := \pi(m^{\alpha}) \delta(\pi(m^{\beta}))$ .

In the following, we note

$$\bar{A} := \bar{A}_{\alpha}^{\beta} M_{\beta}^{\alpha}$$

so for any  $p \in \mathbb{N}$ ,  $\oint \bar{A}|\mathcal{D}|^{-p} = \overline{\oint A|\mathcal{D}|^{-p}}$ .

This presentation of one-forms is not unique modulo  $OP^{-\infty}$  since, as we will see in section 5,  $F = \sum_i x_i dy_i$  where  $x_i, y_i \in \mathcal{A}$ , thus for any generator  $z$ ,  $[F, z] = \sum_i x_i d(y_i z) - x_i y_i dz - z x_i dy_i = 0 \bmod OP^{-\infty}$ . We do not know however if this presentation is unique when the  $OP^{-\infty}$  part is taken into account.

The  $\delta$ -one-forms  $M_{\beta}^{\alpha}$  are said to be *canonical*. Any product of  $n$  canonical  $\delta$ -one forms, where  $n \in \mathbb{N}^*$ , is called a *canonical  $\delta^n$ -one-form*. Thus, if  $A$  is a  $\delta$ -one-form,  $A^n = (A^n)_{\bar{\alpha}}^{\bar{\beta}} M_{\beta}^{\bar{\alpha}}$  where  $\bar{\alpha} = (\alpha, \alpha', \dots, \alpha^{(n-1)})$ ,  $\bar{\beta} = (\beta, \beta', \dots, \beta^{(n-1)})$  are in  $\mathbb{Z}^n \times \mathbb{N}^n \times \mathbb{N}^n$ ,  $(A^n)_{\bar{\alpha}}^{\bar{\beta}} := A_{\alpha}^{\beta} \cdots A_{\alpha^{(n-1)}}^{\beta^{(n-1)}}$  and  $M_{\beta}^{\bar{\alpha}}$  is the canonical  $\delta^n$ -one form equals to  $M_{\beta}^{\alpha} \cdots M_{\beta^{(n-1)}}^{\alpha^{(n-1)}}$ .



**Definition 4.5.** A canonical  $\delta^n$ -one-form is *a-balanced* if it is of the form

$$a^{\alpha_1} \delta(a^{\beta_1}) \cdots a^{\alpha_1^{(n-1)}} \delta(a^{\beta_1^{(n-1)}})$$

where  $\sum_{i=0}^{n-1} \alpha_1^{(i)} + \beta_1^{(i)} = 0$ .

For any  $\delta$ -one-form  $A$ , the *a-balanced* components of  $A^n$  are noted  $B_a(A^n)_{\bar{\alpha}}^{\bar{\beta}}$ .

Note that

$$B_a(A)_{\bar{\alpha}}^{\bar{\beta}} = A_{-\beta_1 00}^{\beta_1 00} \delta_{\alpha_1 + \beta_1, 0} \delta_{\alpha_2 + \alpha_3 + \beta_2 + \beta_3, 0}.$$

**Definition 4.6.** A canonical  $\delta^n$ -one-form is *balanced* if it is of the form

$$m^\alpha \delta(m^\beta) \cdots m^{\alpha^{(n-1)}} \delta(m^{\beta^{(n-1)}})$$

where  $\sum_{i=0}^{n-1} \alpha_1^{(i)} + \beta_1^{(i)} = 0$  and  $\sum_{i=0}^{n-1} \alpha_2^{(i)} + \beta_2^{(i)} = \sum_{i=0}^{n-1} \alpha_3^{(i)} + \beta_3^{(i)}$ .

For any  $\delta$ -one-form  $A$ , the *balanced* components of  $A^n$  are noted  $B(A^n)_{\bar{\alpha}}^{\bar{\beta}}$ .

Note that

$$B(A)_{\bar{\alpha}}^{\bar{\beta}} = A_{-\beta_1 \beta_2 \alpha_3}^{\beta_1 \beta_2 \beta_3} \delta_{\alpha_1 + \beta_1, 0} \delta_{\alpha_2 + \beta_2, \alpha_3 + \beta_3}.$$

As we will show, a contribution to the  $k^{\text{th}}$ -coefficient in the spectral action, is only brought by one-forms  $\mathbb{A}$  such that  $A^k$  is balanced (and even *a-balanced* in the case  $k = 1$ ).

Note also that if  $A$  is balanced, then  $A^k$  for  $k \geq 1$  is also balanced, whereas the converse is false.

#### 4.4 The reality operator $J$ on $SU_q(2)$

For any  $n, p \in \mathbb{N}$ ,

$$\begin{aligned} q_n &:= \sqrt{1 - q^{2n}}, & q_{-n} &:= 0 \text{ if } n > 0, \\ q_{n,p}^\uparrow &:= q_{n+1} \cdots q_{n+p}, & q_{n,p}^\downarrow &:= q_n \cdots q_{n-(p-1)}, \end{aligned}$$

with the convention  $q_{n,0}^\uparrow = q_{n,0}^\downarrow := 1$ . Thus, we have the relations

$$\begin{aligned} \pi_\pm(a^p) \varepsilon_n &= q_{n,p}^\uparrow \varepsilon_{n+p}, & \pi_\pm(a^{*p}) \varepsilon_n &= q_{n,p}^\downarrow \varepsilon_{n-p}, \\ \pi_\pm(b^p) \varepsilon_n &= (\pm q^n)^p \varepsilon_n, & \pi_\pm(b^{*p}) \varepsilon_n &= (\pm q^n)^p \varepsilon_n, \end{aligned}$$

where  $\varepsilon_k := 0$  if  $k < 0$ .

The sign of  $x \in \mathbb{R}$  is noted  $\eta_x$ . By convention,  $a_j := a$ ,  $a_{\pm,j} := a_\pm$  if  $j \geq 0$  and  $a_j := a^*$ ,  $a_{\pm,j} := a_\pm^*$  if  $j < 0$ . Note that, with convention

$$q_{n,p}^{\uparrow \alpha_1} := q_{n,p}^\uparrow \text{ if } \alpha_1 > 0, \quad q_{n,p}^{\uparrow \alpha_1} := q_{n,p}^\downarrow \text{ if } \alpha_1 < 0, \text{ and } q_{n,p}^{\uparrow 0} := 1,$$

we have for any  $\alpha_1 \in \mathbb{Z}$  and  $p \leq \alpha_1$ ,  $\pi_\pm(a_{\alpha_1}^p) \varepsilon_n = q_{n,p}^{\uparrow \alpha_1} \varepsilon_{n+\eta_{\alpha_1} p}$ .

Recall that the reality operator  $J$  is defined by

$$J v_{m,l}^{j\uparrow} = i^{2(m+l)-1} v_{2j-m, 2j+1-l}^{j\uparrow}, \quad J v_{m,l}^{j\downarrow} = i^{-2(m+l)+1} v_{2j-m, 2j-1-l}^{j\downarrow},$$

thus the real conjugate operators

$$\hat{a}_\pm := J a_\pm J^{-1}, \quad \hat{b}_\pm := J b_\pm J^{-1}$$

satisfy

$$\begin{aligned}\widehat{a}_+ v_{m,l}^j &:= -q_{2j+1-m} \begin{pmatrix} q_{2j+2-l} & 0 \\ 0 & q_{2j-l} \end{pmatrix} v_{m,l}^{j+}, & \widehat{a}_- v_{m,l}^j &:= -q^{2j-m} \begin{pmatrix} q^{2j+2-l} & 0 \\ 0 & q^{2j-l} \end{pmatrix} v_{m-1,l-1}^{j-}, \\ \widehat{b}_+ v_{m,l}^j &:= q_{2j+1-m} \begin{pmatrix} q^{2j+1-l} & 0 \\ 0 & q^{2j-1-l} \end{pmatrix} v_{m,l+1}^{j+}, & \widehat{b}_- v_{m,l}^j &:= -q^{2j-m} \begin{pmatrix} q_{2j+1-l} & 0 \\ 0 & q_{2j-1-l} \end{pmatrix} v_{m-1,l}^{j-}.\end{aligned}$$

So the real conjugate operator behave differently on the up and down part of the Hilbert space. The difference comes from the fact that the index  $l$  is not treated uniformly by  $J$  on up and down parts.

We note  $\widehat{X}$  the algebra generated by  $\{\widehat{a}_\pm, \widehat{b}_\pm\}$ ,  $\widetilde{X}$  the algebra generated by  $\{a_\pm, b_\pm, \widehat{a}_\pm, \widehat{b}_\pm\}$  and  $\mathcal{H}' := \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$  and we construct two  $*$ -representations  $\widehat{\pi}_\pm$  of  $\mathcal{A}$ :

The representation  $\widehat{\pi}_+$  gives bounded operators on  $\mathcal{H}'$  while  $\widehat{\pi}_-$  represents  $\mathcal{A}$  into  $\mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2)$ .

The representation  $\widehat{\pi}_+$  is defined on the generators by:

$$\widehat{\pi}_+(a) \varepsilon_m \otimes \varepsilon_{2j} := q_{2j+1-m} \varepsilon_m \otimes \varepsilon_{2j+1}, \quad \widehat{\pi}_+(b) \varepsilon_m \otimes \varepsilon_{2j} := -q^{2j-m} \varepsilon_{m+1} \otimes \varepsilon_{2j+1}$$

while  $\widehat{\pi}_-$  is defined by:

$$\begin{aligned}\widehat{\pi}_-(a) \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &:= -q_{2j+1\pm 1-l} \varepsilon_l \otimes \varepsilon_{2j+1} \otimes \varepsilon_{\uparrow\downarrow}, \\ \widehat{\pi}_-(b) \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &:= -q^{2j\pm 1-l} \varepsilon_{l+1} \otimes \varepsilon_{2j+1} \otimes \varepsilon_{\uparrow\downarrow},\end{aligned}$$

where  $\varepsilon_{\uparrow\downarrow}$  is the canonical basis of  $\mathbb{C}^2$  and the  $+$  in  $\pm$  corresponds to  $\uparrow$  in  $\uparrow\downarrow$ .

The link between  $\widehat{\pi}_\pm$  and  $\pi_\pm$  which explains the notations about these intermediate objects and the fact that  $\widehat{\pi}_\pm$  are representations on different Hilbert spaces, is in the parallel between equations (30), (35) and (36).

Let us give immediately a few properties ( $x_\beta$  equals  $x$  if the sign  $\beta$  is positive and equals  $x^*$  otherwise)

$$\begin{aligned}\widehat{\pi}_+(a_\beta)^p \varepsilon_m \otimes \varepsilon_{2j} &= q_{2j-m,p}^{\uparrow\beta} \varepsilon_m \otimes \varepsilon_{2j+\eta_\beta p}, \\ \widehat{\pi}_-(a_\beta)^p \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &= (-1)^p q_{2j\pm 1-l,p}^{\uparrow\beta} \varepsilon_l \otimes \varepsilon_{2j+\eta_\beta p} \otimes \varepsilon_{\uparrow\downarrow}, \\ \widehat{\pi}_+(b_\beta)^p \varepsilon_m \otimes \varepsilon_{2j} &= (-1)^p q^{(2j-m)p} \varepsilon_{m+\eta_\beta p} \otimes \varepsilon_{2j+\eta_\beta p}, \\ \widehat{\pi}_-(b_\beta)^p \varepsilon_l \otimes \varepsilon_{2j} \otimes \varepsilon_{\uparrow\downarrow} &= (-1)^p q^{(2j\pm 1-l)p} \varepsilon_{l+\eta_\beta p} \otimes \varepsilon_{2j+\eta_\beta p} \otimes \varepsilon_{\uparrow\downarrow}.\end{aligned}$$

Note that the  $\widehat{\pi}_\pm$  representations still contain the shift information, contrary to representations  $\pi_\pm$ . Moreover,  $\widehat{\pi}_\pm(b) \neq \widehat{\pi}_\pm(b^*)$  while  $\pi_\pm(b) = \pi_\pm(b^*)$ .

The operators  $\widehat{a}_\pm, \widehat{b}_\pm$  are coded on  $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$  as the correspondence

$$\begin{aligned}\widehat{a}_+ &\longleftrightarrow \widehat{\pi}_+(a) \otimes \widehat{\pi}_-(a), & \widehat{a}_- &\longleftrightarrow -q \widehat{\pi}_+(b^*) \otimes \widehat{\pi}_-(b^*), \\ \widehat{b}_+ &\longleftrightarrow -\widehat{\pi}_+(a) \otimes \widehat{\pi}_-(b), & \widehat{b}_- &\longleftrightarrow -\widehat{\pi}_+(b^*) \otimes \widehat{\pi}_-(a^*).\end{aligned}\tag{35}$$

We now set the following extension to  $\mathcal{B}(\mathcal{H}')$  of  $\pi_+$  and to  $\mathcal{B}(\mathcal{H}' \otimes \mathbb{C}^2)$  of  $\pi_-$  by

$$\begin{aligned}\pi'_+(a) &:= \pi_+(a) \otimes V, & \pi'_+(b) &:= \pi_+(b) \otimes V \quad (V \text{ is the shift of } \ell^2(\mathbb{Z})), \\ \pi'_-(a) &:= \pi_-(a) \otimes V \otimes 1_2, & \pi'_-(b) &:= \pi_-(b) \otimes V \otimes 1_2.\end{aligned}$$

So, we can define a canonical algebra morphism  $\widetilde{\rho}$  from  $\widetilde{X}$  into the bounded operators on  $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$ . This morphism is defined on the generators part  $\{\widehat{a}_\pm, \widehat{b}_\pm\}$  of  $\widetilde{X}$  by preceding correspondence and on the generators part  $\{a_\pm, b_\pm\}$  by –see (30):

$$\begin{aligned}a_+ &\longleftrightarrow \pi'_+(a) \otimes \pi'_-(a), & a_- &\longleftrightarrow -q \pi'_+(b^*) \otimes \pi'_-(b^*), \\ b_+ &\longleftrightarrow -\pi'_+(a) \otimes \pi'_-(b), & b_- &\longleftrightarrow -\pi'_+(b^*) \otimes \pi'_-(a^*).\end{aligned}\tag{36}$$

We note  $S$  the canonical surjection from  $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$  onto  $\mathcal{H}$ . This surjection is associated to the parameters restrictions on  $m, j, l, j'$ . In particular, the index  $j'$  associated to the second  $\ell^2(\mathbb{N})$  in  $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$  is set to be equal to  $j$ . Any vector in  $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$  not satisfying these restrictions is sent to 0 in  $\mathcal{H}$ .

Denote by  $I$  the canonical injection of  $\mathcal{H}$  into  $\mathcal{H}' \otimes \mathcal{H}' \otimes \mathbb{C}^2$  (the index  $j$  is doubled). Thus,  $S\tilde{\rho}(\cdot)I$  is the identity on  $\tilde{X}$ .

In the computation of residues of  $\zeta_{\mathcal{D}}^T$  functions, we can therefore replace the operator  $T$  by  $S\tilde{\rho}(T)I$ .

We now extend  $\tau_0$  on  $\pi'_{\pm}(\mathcal{A})\hat{\pi}_{\pm}(\mathcal{A})$ : For  $x, y \in \mathcal{A}$ , we set

$$\begin{aligned}\mathrm{Tr}_N(\pi'_+(x)\hat{\pi}_+(y)) &:= \sum_{m=0}^N \langle \varepsilon_m \otimes \varepsilon_N, \pi'_+(x)\hat{\pi}_+(y) \varepsilon_m \otimes \varepsilon_N \rangle, \\ \mathrm{Tr}_N^{\uparrow}(\pi'_-(x)\hat{\pi}_-(y)) &:= \sum_{l=0}^N \langle \varepsilon_l \otimes \varepsilon_{N-1} \otimes \varepsilon_{\uparrow}, \pi'_-(x)\hat{\pi}_-(y) \varepsilon_l \otimes \varepsilon_{N-1} \otimes \varepsilon_{\uparrow} \rangle, \\ \mathrm{Tr}_N^{\downarrow}(\pi'_-(x)\hat{\pi}_-(y)) &:= \sum_{l=0}^N \langle \varepsilon_l \otimes \varepsilon_{N+1} \otimes \varepsilon_{\downarrow}, \pi'_-(x)\hat{\pi}_-(y) \varepsilon_l \otimes \varepsilon_{N+1} \otimes \varepsilon_{\downarrow} \rangle.\end{aligned}$$

Actually, a computation on monomials of  $\mathcal{A}$  shows that  $\mathrm{Tr}_N^{\downarrow}(\pi'_-(x)\hat{\pi}_-(y)) = \mathrm{Tr}_N^{\uparrow}(\pi'_-(x)\hat{\pi}_-(y))$ . For convenience, we shall note  $\mathrm{Tr}_N(\pi'_-(x)\hat{\pi}_-(y))$  this functional.

**Lemma 4.7.** *Let  $x, y \in \mathcal{A}$ . Then,*

(i)  $\tau_0(\pi'_{\pm}(x)\hat{\pi}_{\pm}(y)) := \lim_{N \rightarrow \infty} U_N$  exists where

$$U_N := \mathrm{Tr}_N(\pi'_{\pm}(x)\hat{\pi}_{\pm}(y)) - (N+1)\tau_1(\pi_{\pm}(x))\tau_1(\pi_{\pm}(y)).$$

(ii)  $U_N = \tau_0(\pi'_{\pm}(x)\hat{\pi}_{\pm}(y)) + \mathcal{O}(N^{-k})$  for all  $k > 0$ .

*Proof.* (i) We can suppose that  $x$  and  $y$  are monomials, since the result will follow by linearity. We will give a proof for the case of the  $\pi_+$  representations, the case  $\pi_-$  being similar, with minor changes.

We have  $\hat{\pi}_+(y) = (\hat{\pi}_+ a_{\beta_1})^{|\beta_1|} (\hat{\pi}_+ b)^{\beta_2} (\hat{\pi}_+ b^*)^{\beta_3}$ . A computation gives

$$\hat{\pi}_+(y) \varepsilon_m \otimes \varepsilon_{2j} = (-1)^{\beta_2+\beta_3} q^{(2j-m)(\beta_2+\beta_3)} q_{2j-m, |\beta_1|}^{\uparrow \beta_1} \varepsilon_{m-\beta_3+\beta_2} \otimes \varepsilon_{2j-\beta_3+\beta_2+\beta_1}$$

and with the notation  $t_{2j,m} := \langle \varepsilon_m \otimes \varepsilon_{2j}, \pi'_{\pm}(x)\hat{\pi}_{\pm}(y) \varepsilon_m \otimes \varepsilon_{2j} \rangle$  and  $T_{2j} := \sum_{m=0}^{2j} t_{2j,m}$ , we get

$$\begin{aligned}t_{2j,m} &= (-1)^{\beta_2+\beta_3} q^{(2j-m)(\beta_2+\beta_3)} q_{2j-m, |\beta_1|}^{\uparrow \beta_1} q_{m-\beta_3+\beta_2, |\alpha_1|}^{\uparrow \alpha_1} q^{(m+\beta_2-\beta_3)(\alpha_2+\alpha_3)} \delta_{\alpha_1+\beta_2-\beta_3, 0} \\ &\quad \times \delta_{-\alpha_3+\alpha_2+\beta_1, 0} \\ &= (-1)^{\alpha_1} q^{(2j-m)(\beta_2+\beta_3)+(m-\alpha_1)(\alpha_2+\alpha_3)} q_{2j-m, |\beta_1|}^{\uparrow \beta_1} q_{m-\alpha_1, |\alpha_1|}^{\uparrow \alpha_1} \delta_{\alpha_1+\beta_2-\beta_3, 0} \delta_{\alpha_2-\alpha_3+\beta_1, 0} \\ &=: f_{\alpha, \beta} q^{2j\lambda} t'_{2j,m} =: f_{\alpha, \beta} q^{2j\kappa} t''_{2j, 2j-m}\end{aligned}$$

where

$$t'_{2j,m} := q^{m(\kappa-\lambda)} q_{2j-m, |\beta_1|}^{\uparrow \beta_1} q_{m-\alpha_1, |\alpha_1|}^{\uparrow \alpha_1}, \quad (37)$$

$$t''_{2j,m} := q^{m(\lambda-\kappa)} q_{m, |\beta_1|}^{\uparrow \beta_1} q_{2j-m-\alpha_1, |\alpha_1|}^{\uparrow \alpha_1}, \quad (38)$$

with  $\lambda := \beta_2 + \beta_3 \geq 0$  and  $\kappa := \alpha_2 + \alpha_3 \geq 0$ . We will now prove that if  $\lambda \neq \kappa$ , then  $(T_{2j})$  is a convergent sequence. Suppose  $\kappa > \lambda$ . Let us note  $U'_{2j} := \sum_{m=0}^{2j} t'_{2j,m}$ . Since the  $t'_{2j,m}$  are positive and  $t'_{2j+1,m} \geq t'_{2j,m}$  for all  $j, m$ ,  $U'_{2j}$  is an increasing real sequence. The estimate

$$U'_{2j} \leq \sum_{m=0}^{2j} q^{m(\kappa-\lambda)} \leq \frac{1}{1-q^{\kappa-\lambda}} < \infty$$

proves then that  $U'_{2j}$  is a convergent sequence. With  $T_{2j} = f_{\alpha,\beta} q^{2j\lambda} U'_{2j}$ , we obtain our result. Suppose now that  $\lambda > \kappa$ . Let us note  $U''_{2j} := \sum_{m=0}^{2j} t''_{2j,m}$ . Since the  $t''_{2j,m}$  are positive and  $t''_{2j+1,m} \geq t''_{2j,m}$  for all  $j, m$ ,  $U''_{2j}$  is an increasing real sequence. The estimate

$$U''_{2j} \leq \sum_{m=0}^{2j} q^{m(\lambda-\kappa)} \leq \frac{1}{1-q^{\lambda-\kappa}} < \infty$$

proves then that  $U''_{2j}$  is a convergent sequence. With  $T_{2j} = f_{\alpha,\beta} q^{2j\kappa} U''_{2j}$ , we have again our result. Moreover, note that if  $\lambda$  and  $\kappa$  are both different from zero, the limit of  $(T_{2j})$  is zero and more precisely,

$$T_{2j} = \mathcal{O}(q^{2j\lambda}) \text{ if } \kappa > \lambda > 0, \quad (39)$$

$$T_{2j} = \mathcal{O}(q^{2j\kappa}) \text{ if } \lambda > \kappa > 0. \quad (40)$$

Suppose now that  $\lambda = \kappa \neq 0$ . In that case,  $(T_{2j})$  also converges rapidly to zero. Indeed, let us fix  $q < \varepsilon < 1$ . we have  $\varepsilon^{-2j\lambda} T_{2j} = \sum_{m=0}^{2j} c_m d_{2j-m} = c * d(2j)$  where  $c_m := f_{\alpha,\beta} (q/\varepsilon)^{\lambda m} q_{m-\alpha_1, |\alpha_1|}^{\uparrow_{\alpha_1}}$  and  $d_m := (q/\varepsilon)^{\lambda m} q_{m, |\beta_1|}^{\uparrow_{\beta_1}}$ . Since both  $\sum_m c_m$  and  $\sum_m d_m$  are absolutely convergent series, their Cauchy product  $\sum_{2j} \varepsilon^{-2j\lambda} T_{2j}$  is convergent. In particular,  $\lim_{j \rightarrow \infty} \varepsilon^{-2j\lambda} T_{2j} = 0$ , and

$$T_{2j} = \mathcal{O}(\varepsilon^{2j\lambda}). \quad (41)$$

Finally,  $T_{2j}$  has a finite limit in all cases except possibly when  $\lambda = \kappa = 0$ , which is the case when  $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$ . In that case,  $t_{2j,m} = 1$ .

A straightforward computation gives  $\tau_1(\pi_{\pm}(x)) \tau_1(\pi_{\pm}(y)) = \delta_{\alpha_1,0} \delta_{\beta_1,0} \delta_{\alpha_2,0} \delta_{\beta_2,0} \delta_{\alpha_3,0} \delta_{\beta_3,0}$ .

Thus,

$$U_{2j} = T_{2j} - (2j+1) \delta_{\alpha_1,0} \delta_{\beta_1,0} \delta_{\alpha_2,0} \delta_{\beta_2,0} \delta_{\alpha_3,0} \delta_{\beta_3,0}$$

has always a finite limit when  $j \rightarrow \infty$ .

(ii) The result is clear if  $\lambda = \kappa = 0$  (in that case  $U_N = \tau_0 = 0$ ). Suppose  $\lambda$  or  $\kappa$  is not zero. In that case  $U_{2j} = T_{2j}$ . By (40), (39) and (41), we see that if  $\lambda > \kappa > 0$  or  $\kappa > \lambda > 0$  or  $\kappa = \lambda$ ,  $(T_{2j})$  converges to 0 with a rate in  $\mathcal{O}(\varepsilon^{2j\alpha})$  where  $\alpha > 0$  and  $q \leq \varepsilon < 1$ . Thus, it only remains to check the cases  $(\kappa > 0, \lambda = 0)$  and  $(\kappa = 0, \lambda > 0)$ . In the first one, we get from (37),  $U_{2j} = f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa} q_{2j-m, |\beta_1|}^{\uparrow_{\beta_1}}$ . If  $\beta_1 = 0$ , we are done.

Suppose  $\beta_1 > 0$ . We have  $q_{2j-m, |\beta_1|}^{\uparrow_{\beta_1}} = \sum_{p=0}^{\infty} l_p q^{r_p} q^{2|p|_1(2j-m)}$  where  $p = (p_1, \dots, p_{\beta_1})$  and  $l_p = (-1)^{|p|_1} \left(\frac{1}{p}\right)$ ,  $r_p := 2p_1 + \dots + 2\beta_1 p_{\beta_1}$ . Thus, cutting the sum in two, we get, noting  $L_{2j} := f_{\alpha,\beta} \sum_{m=0}^{2j} q^{m\kappa}$ ,

$$U_{2j} - L_{2j} = f_{\alpha,\beta} \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{q^{4|p|_1 j - q^{(2j+1)\kappa - 2|p|_1}}}{1 - q^{\kappa - 2|p|_1}} + f_{\alpha,\beta} \sum_{0 \neq |p|_1 \leq \kappa/2} l_p q^{r_p} q^{4|p|_1 j} \sum_{m=0}^{2j} q^{m(\kappa - 2|p|_1)}.$$

Since  $\sum_{0 \neq |p|_1 \leq \kappa/2} l_p q^{r_p} q^{4|p|_1 j} \sum_{m=0}^{2j} q^{m(\kappa-2|p|_1)}$  is in  $\mathcal{O}_{j \rightarrow \infty}(jq^{4j})$ , we have, modulo a rapidly decreasing sequence,

$$U_{2j} - L_{2j} \sim f_{\alpha, \beta} \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{q^{4|p|_1 j - q^{(2j+1)\kappa-2|p|_1}}}{1 - q^{\kappa-2|p|_1}} =: f_{\alpha, \beta} q^{2\kappa j} V_{2j}$$

with

$$V_{2j} = \sum_{|p|_1 > \kappa/2} l_p q^{r_p} \frac{1 - q^{(2|p|_1 - \kappa)(2j+1)}}{1 - q^{2|p|_1 - \kappa}} = \sum_{|p|_1 > \kappa/2} \sum_{m=0}^{2j} l_p q^{r_p} q^{(2|p|_1 - \kappa)m}.$$

The family  $v_{m,p} := (l_p q^{r_p} q^{(2|p|_1 - \kappa)m})_{(p,m) \in I}$ , where  $I = \{(p,m) \in \mathbb{N}^{\beta_1} \times \mathbb{N} : |p|_1 > \kappa/2\}$  is (absolutely) summable. Indeed  $|v_{m,p}| \leq |l_p| q^{r_p} q^m$  so  $|v_{m,p}|$  is summable as the product of two summable families. As a consequence,  $\lim_{j \rightarrow \infty} V_{2j}$  exists and is finite, which proves that  $(q^{2\kappa j} V_{2j})$ , and thus  $(U_{2j} - L_{2j})$  converge rapidly to 0.

Suppose now that  $\beta_1 < 0$ . In that case,  $q_{2j-m, |\beta_1|}^{\uparrow \beta_1} = q_{2j-m, |\beta_1|}^{\downarrow} = q_{2j-(m+|\beta_1|), |\beta_1|}^{\uparrow}$  and by (37), we get  $U_{2j} = f_{\alpha, \beta} \sum_{m=0}^{2j} q^{m\kappa} q_{2j-(m+|\beta_1|), |\beta_1|}^{\uparrow} = f_{\alpha, \beta} q^{-|\beta_1|\kappa} \sum_{m=|\beta_1|}^{2j+|\beta_1|} q^{m\kappa} q_{2j-m, |\beta_1|}^{\uparrow}$ , so the same arguments as in case  $\beta_1 > 0$  apply here, the summation on  $m$  simply shifted of  $|\beta_1|$ .

The same proof can be applied for the other case ( $\kappa = 0, \lambda > 0$ ). This time, we only need to use (38) instead of (37) and the preceding arguments follow by replacing  $\kappa$  by  $\lambda$  and  $\beta_1$  by  $\alpha_1$ .  $\square$

**Remark 4.8.** *Contrary to the preceding  $\tau_0$ , the new functional contains the shift information. In particular, it filters the parts of nonzero degree.*

If  $T \in X\hat{X}$ ,  $\tilde{\rho}(T) \in \pi_+(\mathcal{A})\hat{\pi}_+(\mathcal{A}) \otimes \pi_-(\mathcal{A})\hat{\pi}_-(\mathcal{A})$ .

For notational convenience, we define  $\tau_1$  on  $\pi'_\pm(\mathcal{A})\hat{\pi}_\pm(\mathcal{A})$  as

$$\tau_1(\pi'_\pm(x)\hat{\pi}_\pm(y)) := \tau_1(\pi_\pm(x)) \tau_1(\pi_\pm(y)).$$

In the following, the symbol  $\sim_e$  means equals modulo a entire function.

**Theorem 4.9.** *Let  $T \in X\hat{X}$ . Then*

- (i)  $\zeta_D^T(s) \sim_e 2(\tau_1 \otimes \tau_1)(\tilde{\rho}(T)) \zeta(s-2) + 2(\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0)(\tilde{\rho}(T)) \zeta(s-1) + 2(\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1)(\tilde{\rho}(T)) \zeta(s),$
- (ii)  $\oint T|\mathcal{D}|^{-3} = 2(\tau_1 \otimes \tau_1)(\tilde{\rho}(T)),$
- (iii)  $\oint T|\mathcal{D}|^{-2} = 2(\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0)(\tilde{\rho}(T)),$
- (iv)  $\oint T|\mathcal{D}|^{-1} = 2(\tau_0 \otimes \tau_0 - \frac{1}{2}\tau_1 \otimes \tau_1)(\tilde{\rho}(T)).$

*Proof.* (i) Since  $T \in X\hat{X}$ ,  $\tilde{\rho}(T)$  is a linear combination of terms like  $\pi'_+(x)\hat{\pi}_+(y) \otimes \pi'_-(z)\hat{\pi}_-(t)$ , where  $x, y, z, t \in \mathcal{A}$ . Such a term is noted in the following  $T_+ \otimes T_-$ . Linear combination of these term is implicit. With the shortcut  $T_{c_1, \dots, c_p} := \langle \varepsilon_{c_1} \otimes \dots \otimes \varepsilon_{c_p}, T \varepsilon_{c_1} \otimes \dots \otimes \varepsilon_{c_p} \rangle$ , recalling that

$v_{m,l}^{j,\downarrow}$  is 0 when  $j = 0$ , or  $l \geq 2j$ , we get

$$\begin{aligned}
\zeta_D^T(s) &= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle (v_{m,l}^{j,\uparrow}), S\tilde{\rho}(T)I(v_{m,l}^{j,\uparrow}) \rangle d_{j+}^{-s} + \langle (v_{m,l}^{j,\downarrow}), S\tilde{\rho}(T)I(v_{m,l}^{j,\downarrow}) \rangle d_j^{-s} \\
&= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \tilde{\rho}(T)_{m,2j,l,2j,\uparrow} d_{j+}^{-s} + \sum_{2j=1}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j-1} \tilde{\rho}(T)_{m,2j,l,2j,\downarrow} d_j^{-s} \\
&= \sum_{2j=0}^{\infty} (\text{Tr}_{2j}(T_+) \text{Tr}_{2j+1}^{\uparrow}(T_-) + \text{Tr}_{2j+1}(T_+) \text{Tr}_{2j}^{\downarrow}(T_-)) d_{j+}^{-s}.
\end{aligned}$$

By Lemma 4.7 (ii), for all  $k > 0$ ,

$$\begin{aligned}
\text{Tr}_{2j}(T_{\pm}) &= (2j + \frac{3}{2})\tau_1(T_{\pm}) + \tau_0(T_{\pm}) - \frac{1}{2}\tau_1(T_{\pm}) + \mathcal{O}((2j)^{-k}), \\
\text{Tr}_{2j+1}(T_{\pm}) &= (2j + \frac{3}{2})\tau_1(T_{\pm}) + \tau_0(T_{\pm}) + \frac{1}{2}\tau_1(T_{\pm}) + \mathcal{O}((2j)^{-k}).
\end{aligned}$$

The result follows by noting that the difference of the Hurwitz zeta function  $\zeta(s, \frac{3}{2})$  and Riemann zeta function  $\zeta(s)$  is an entire function.

(ii, iii, iv) are direct consequences of (i).  $\square$

#### 4.5 The smooth algebra $C^\infty(SU_q(2))$

In [13, 38], the smooth algebra  $C^\infty(SU_q(2))$  is defined by pulling back the smooth structure  $C^\infty(D_{q^\pm}^2)$  into the  $C^*$ -algebra generated by  $\mathcal{A}$ , through the morphism  $\rho$  and the application  $\lambda$  (the compression which gives an operator on  $\mathcal{H}$  from an operator on  $l^2(\mathbb{N}) \otimes l^2(\mathbb{N}) \otimes l^2(\mathbb{Z}) \otimes \mathbb{C}^2$ ). The important point is that with [13, Lemma 2, p. 69], this algebra is stable by holomorphic calculus. By defining  $\rho := \tilde{\rho} \circ c$  and  $\lambda(\cdot) := S(\cdot)I$ , the same lemma (with same notation) can be applied to our setting, with  $c := \pi(x) \mapsto \underline{\pi}(x)$  and

$$\mathcal{C} := C^\infty(D_{q^+}^2) \otimes C^\infty(S^1) \otimes C^\infty(D_{q^+}^2) \otimes C^\infty(S^1) \otimes \mathcal{M}_2(\mathbb{C})$$

as algebra stable by holomorphic calculus containing the image of  $\tilde{\rho}$ . Here, we use Schwartz sequences to define the smooth structures. We finally obtain  $C^\infty(SU_q(2))$  with real structure as a subalgebra stable by holomorphic calculus of the  $C^*$ -algebra generated by  $\pi(\mathcal{A}) \cup J\pi(\mathcal{A})J^{-1}$  and containing  $\pi(\mathcal{A}) \cup J\pi(\mathcal{A})J^{-1}$ .

**Corollary 4.10.** *The dimension spectrum of the real spectral triple  $(C^\infty(SU_q(2)), \mathcal{H}, D)$  is simple and given by  $\{1, 2, 3\}$ . Its  $KO$ -dimension is 3.*

*Proof.* Since  $F$  commutes with  $\underline{\pi}(\mathcal{A})$ , the pseudodifferential operators of order 0 (without the real structure and in the sense of [22]) are exactly (modulo  $OP^{-\infty}$ ) the operators in  $\mathcal{B} + \mathcal{B}F$ . From Theorem 3.4 we see that the dimension spectrum of  $SU_q(2)$  without taking into account the reality operator  $J$  is  $\{1, 2, 3\}$ , in other words, the possible poles of  $\zeta_D^b : s \mapsto \text{Tr}(bF^\varepsilon|\mathcal{D}|^{-s})$  (with  $\varepsilon \in \{0, 1\}$ ,  $b \in \mathcal{B}$ ) are in  $\{1, 2, 3\}$ . Theorem 4.9 (i) shows that the possible poles are still  $\{1, 2, 3\}$  when we take into account the real structure of  $SU_q(2)$ , that is to say, when  $\mathcal{B}$  is enlarged to  $\mathcal{B}J\mathcal{B}J^{-1}$ . Indeed, any element of  $\mathcal{B}J\mathcal{B}J^{-1}$  is in  $X\hat{X}$  and it is clear from the preceding proof that adding  $F$  in the previous zeta function do not add any pole to  $\{1, 2, 3\}$ . All arguments goes true from the polynomial algebra  $\mathcal{A}(SU_q(2))$  to the smooth pre- $C^*$ -algebra  $C^\infty(SU_q(2))$ .

$KO$ -dimension refers just to  $J^2 = -1$  and  $\mathcal{D}J = J\mathcal{D}$  since there is no chirality because spectral dimension is 3.  $\square$

## 4.6 Noncommutative integrals with reality operator and one-forms on $SU_q(2)$

The goal of this section is to obtain the following suppression of  $J$ :

**Theorem 4.11.** *Let  $A$  and  $B$  be  $\delta$ -one-forms. Then*

$$\begin{aligned}
(i) \quad & \oint A J B J^{-1} |\mathcal{D}|^{-3} = \frac{1}{2} \oint A |\mathcal{D}|^{-3} \overline{\oint B |\mathcal{D}|^{-3}}, \\
(ii) \quad & \oint A J B J^{-1} |\mathcal{D}|^{-2} = \frac{1}{2} \oint A |\mathcal{D}|^{-2} \overline{\oint B |\mathcal{D}|^{-3}} + \frac{1}{2} \oint A |\mathcal{D}|^{-3} \overline{\oint B |\mathcal{D}|^{-2}}, \\
(iii) \quad & \oint A^2 J B J^{-1} |\mathcal{D}|^{-3} = \frac{1}{2} \oint A^2 |\mathcal{D}|^{-3} \overline{\oint B |\mathcal{D}|^{-3}}, \\
(iv) \quad & \oint \delta(A) A |\mathcal{D}|^{-3} = \oint \delta(A) J A J^{-1} |\mathcal{D}|^{-3} = 0.
\end{aligned}$$

We gather at the beginning of this section the main notations for technical lemmas which will follow.

For any pair  $(k, p) \in \mathbb{N}^3 \times \mathbb{N}^3$  such that  $k_i \leq |\alpha_i|$ ,  $p_i \leq |\beta_i|$ , where  $\alpha, \beta \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ , we define

$$\begin{aligned}
v_{k,p} &:= g(p) \binom{|\alpha_1|}{k_1}_{q^{2\eta_{\alpha_1}}} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3} \binom{|\beta_1|}{p_1}_{q^{2\eta_{\beta_1}}} \binom{\beta_2}{p_2} \binom{\beta_3}{p_3} (-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3} q^{\sigma_{k,p}}, \\
h_{k,p} &:= \alpha_1 + \alpha_2 - \alpha_3 - 2(\eta_{\alpha_1} k_1 + k_2 - k_3) + g(p), \\
g(p) &:= \beta_1 + \beta_2 - \beta_3 - 2(\eta_{\beta_1} p_1 + p_2 - p_3), \\
\sigma_{k,p} &:= k_1 + p_1 + \sigma_{k,p}^t + \sigma_{k,p}^u, \\
\sigma_{k,p}^t &:= k_1 \widehat{k}_2 - \widehat{k}_3(k_1 + k_2) + \eta_{\beta_1} \widehat{p}_1 |k|_1 + \widehat{p}_2(|k|_1 + p_1) - \widehat{p}_3(|k|_1 + p_1 + p_2), \\
\sigma_{k,p}^u &:= (k_3 + \eta_{\beta_1} \widehat{p}_1 - p_2 + p_3)(k_1 + \widehat{k}_2 + \widehat{k}_3) - k_2(k_1 + \widehat{k}_2) + (p_1 + \widehat{p}_2)(-p_2 + p_3) + \widehat{p}_3 p_3, \\
t_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} a^{\widehat{k}_2} a^{*\widehat{k}_3} a_{\beta_1}^{\widehat{p}_1} a^{\widehat{p}_2} a^{*\widehat{p}_3} b^{|k|_1+|p|_1}, \\
u_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} a^{*\widehat{k}_2} a^{k_3} a_{\beta_1}^{\widehat{p}_1} a^{*p_2} a^{p_3} b^{|\widetilde{k}|_1+|\widetilde{p}|_1}.
\end{aligned}$$

where we used the notation

$$\widehat{k}_i := |\alpha_i| - k_i, \quad \widehat{p}_i := |\beta_i| - p_i,$$

so  $0 \leq \widehat{k}_i \leq |\alpha_i|$ ,  $0 \leq \widehat{p}_i \leq |\beta_i|$ . We will also use the shortcut  $\widetilde{k} := (k_1, \widehat{k}_2, \widehat{k}_3)$ .

For  $\beta_1 \in \mathbb{Z}$  and  $j \in \mathbb{N}$ , we define

$$\begin{aligned}
w_1(\beta_1, j) &:= \sum_{n=0}^{\infty} (q^{2jn} (q_{n,|\beta_1|}^{\uparrow \beta_1})^2 - \delta_{j,0}), \\
w_{\beta}^{\alpha} &:= 2\beta_1 q^{\beta_1(2\alpha_3+\beta_3-\beta_2)} w_1(\beta_1, \alpha_3 + \beta_3).
\end{aligned}$$

We introduce the following notations:

$$\begin{aligned}
q_{k,p,n}^+ &:= q^{n(|k|_1+|p|_1)} q_{n+r_{k,p}^+ - \eta_{\alpha_1} \widehat{k}_1, \widehat{k}_1}^{\uparrow \alpha_1} q_{n-\widehat{k}_3+\eta_{\beta_1} \widehat{p}_1+\widehat{p}_2-\widehat{p}_3, \widehat{k}_2}^{\downarrow} q_{n+\eta_{\beta_1} \widehat{p}_1+\widehat{p}_2-\widehat{p}_3, \widehat{k}_3}^{\downarrow} q_{n+\widehat{p}_2-\widehat{p}_3, \widehat{p}_1}^{\uparrow \beta_1} q_{n-\widehat{p}_3, \widehat{p}_2}^{\uparrow} q_{n, \widehat{p}_3}^{\downarrow}, \\
q_{k,p,n}^- &:= q^{n(|\widetilde{k}|_1+|\widetilde{p}|_1)} q_{n+r_{k,p}^- - \eta_{\alpha_1} \widehat{k}_1, \widehat{k}_1}^{\uparrow \alpha_1} q_{n+k_3+\eta_{\beta_1} \widehat{p}_1-p_2+p_3, k_2}^{\downarrow} q_{n+\eta_{\beta_1} \widehat{p}_1-p_2+p_3, k_3}^{\uparrow} q_{n-p_2+p_3, \widehat{p}_1}^{\uparrow \beta_1} q_{n+p_3, p_2}^{\downarrow} q_{n, p_3}^{\uparrow} \\
&\quad \times (-1)^{|\widetilde{k}|_1+|\widetilde{p}|_1}, \\
r_{k,p}^+ &:= \eta_{\alpha_1} \widehat{k}_1 + \widehat{k}_2 - \widehat{k}_3 + \eta_{\beta_1} \widehat{p}_1 + \widehat{p}_2 - \widehat{p}_3, \\
r_{k,p}^- &:= \eta_{\alpha_1} \widehat{k}_1 - k_2 + k_3 + \eta_{\beta_1} \widehat{p}_1 - p_2 + p_3.
\end{aligned}$$

Thus,  $\pi_+(t_{k,p})\varepsilon_n = q_{k,p,n}^+ \varepsilon_{n+r_{k,p}^+}$  and  $\pi_-(u_{k,p})\varepsilon_n = q_{k,p,n}^- \varepsilon_{n+r_{k,p}^-}$ .

**Lemma 4.12.** *We have*

$$r((M_\beta^\alpha)^\circ) = \sum_{k,p} \delta_{h_{k,p},0} v_{k,p} \pi_+(t_{k,p}) \otimes \pi_-(u_{k,p})$$

where the summation is done on  $k_i, p_i$  in  $\mathbb{N}$  such that  $k_i \leq |\alpha_i|, p_i \leq |\beta_i|$  for  $i \in \{1, 2, 3\}$ .

*Proof.* Since  $\pi(m^\alpha) = (a_+ + a_-)^{\alpha_1} (b_+ + b_-)^{\alpha_2} (b_+^* + b_-^*)^{\alpha_3}$ , with  $v_k := \binom{|\alpha_1|}{k_1}_{q^{2\eta_{\alpha_1}}} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3}$ ,

$$\pi(m^\alpha) = \sum_k v_k c_k \text{ where } c_k := a_{+,\alpha_1}^{|\alpha_1|-k_1} a_{-,\alpha_1}^{k_1} b_+^{\alpha_2-k_2} b_-^{k_2} b_+^{*\alpha_3-k_3} b_-^{*k_3}.$$

By Lemma 3.2 (iii) we see that  $\delta(\pi(m^\beta)) = \sum_p w_p d_p$  where we introduce

$$w_p := \binom{|\beta_1|}{p_1}_{q^{2\eta_{\beta_1}}} \binom{\beta_2}{p_2} \binom{\beta_3}{p_3} \text{ and } d_p := g(p) a_{+,\beta_1}^{|\beta_1|-p_1} a_{-,\beta_1}^{p_1} b_+^{\beta_2-p_2} b_-^{p_2} b_+^{*\beta_3-p_3} b_-^{*p_3}.$$

As a consequence,  $(M_\beta^\alpha)^\circ = \sum_{k,p} \delta_{h(k,p),0} g(p) v_k w_p c_{k,p}$  where

$$c_{k,p} = a_{+,\alpha_1}^{\widehat{k}_1} a_{-,\alpha_1}^{k_1} b_+^{\widehat{k}_2} b_-^{k_2} b_+^{*\widehat{k}_3} b_-^{*k_3} a_{+,\beta_1}^{\widehat{p}_1} a_{-,\beta_1}^{p_1} b_+^{\widehat{p}_2} b_-^{p_2} b_+^{*\widehat{p}_3} b_-^{*p_3} \quad (42)$$

With (42), we get  $r(c_{k,p}) = (-1)^{k_1+p_1+\alpha_2+\alpha_3+\beta_2+\beta_3} q^{k_1+p_1} \pi_+(t'_{k,p}) \otimes \pi_-(u'_{k,p})$  where

$$\begin{aligned} t'_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} b^{k_1} a^{\widehat{k}_2} b^{k_2} a^{*\widehat{k}_3} b^{k_3} a_{\beta_1}^{\widehat{p}_1} b^{p_1} a^{\widehat{p}_2} b^{p_2} a^{*\widehat{p}_3} b^{p_3}, \\ u'_{k,p} &= a_{\alpha_1}^{\widehat{k}_1} b^{k_1} b^{\widehat{k}_2} a^{*k_2} b^{\widehat{k}_3} a^{k_3} a_{\beta_1}^{\widehat{p}_1} b^{p_1} b^{\widehat{p}_2} a^{*p_2} b^{\widehat{p}_3} a^{p_3}. \end{aligned}$$

A recursive use of relation  $ba_j = q^{\eta_j} a_j b$  yields the result.  $\square$

**Lemma 4.13.** *We have*

$$(i) (\tau_1 \otimes \tau_1)(r(M_\beta^\alpha)^\circ) = \beta_1 \delta_{\alpha_1, -\beta_1} \delta_{\alpha_2, 0} \delta_{\alpha_3, 0} \delta_{\beta_2, 0} \delta_{\beta_3, 0}.$$

$$(ii) (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(M_\beta^\alpha)^\circ) = \delta_{\alpha_1, -\beta_1} \delta_{\alpha_2+\beta_2, \alpha_3+\beta_3} w_\beta^\alpha.$$

In particular, if  $A$  is a  $\delta$ -one-form, we have

$$\begin{aligned} \oint A |\mathcal{D}|^{-3} &= 2\beta_1 A_{-\beta_1 00}^{\beta_1 00}, \\ \oint A |\mathcal{D}|^{-2} &= 2w_\beta^\alpha B(A)_\alpha^\beta. \end{aligned}$$

where we implicitly summed on all  $\alpha, \beta$  indices.

*Proof.* (i) Using same notations of Lemma 4.12, we obtain by definition of  $\tau_1$ ,

$$\tau_1(\pi_+(t_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}, \quad (43)$$

$$\tau_1(\pi_-(u_{k,p})) = \delta_{\widetilde{k},0} \delta_{\widetilde{p},0} \delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}. \quad (44)$$

We get  $\tau_1(\pi_+(t_{k,p})) \tau_1(\pi_-(u_{k,p})) = \delta_{k,0} \delta_{p,0} \delta_{\alpha_2,0} \delta_{\alpha_3,0} \delta_{\beta_2,0} \delta_{\beta_3,0} \delta_{\alpha_1, -\beta_1}$ , so Lemma 4.12 gives the result.



(ii) Since  $\pi_+(t_{k,p})\varepsilon_n = q_{k,p,n}^+ \varepsilon_{n+r_{k,p}^+}$  and  $\pi_-(u_{k,p})\varepsilon_n = q_{k,p,n}^- \varepsilon_{n+r_{k,p}^-}$ , we get,

$$\tau_0(\pi_+(t_{k,p})) = \delta_{r_{k,p}^+,0} \sum_{n=0}^{\infty} (q_{k,p,n}^+ - \delta_{k,0} \delta_{p,0} \delta_{\alpha_1+\alpha_2-\alpha_3+\beta_1+\beta_2-\beta_3,0}), \quad (45)$$

$$\tau_0(\pi_-(u_{k,p})) = \delta_{r_{k,p}^-,0} \sum_{n=0}^{\infty} (q_{k,p,n}^- - \delta_{k,0} \delta_{\tilde{p},0} \delta_{\alpha_1-\alpha_2+\alpha_3+\beta_1-\beta_2+\beta_3,0}). \quad (46)$$

With (43) and (46) we get

$$\begin{aligned} \tau_1(\pi_+(t_{k,p})) \tau_0(\pi_-(u_{k,p})) &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} \sum_{n=0}^{\infty} (\delta_{k,0} \delta_{p,0} q_{k,p,n}^- - \delta_{\alpha_3+\beta_3,0}) \\ &= \delta_{k,0} \delta_{p,0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} w_1(\beta_1, \alpha_3 + \beta_3). \end{aligned}$$

Using (44) and (45),

$$\begin{aligned} \tau_0(\pi_+(t_{k,p})) \tau_1(\pi_-(u_{k,p})) &= \delta_{\tilde{k},0} \delta_{\tilde{p},0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} \sum_{n=0}^{\infty} (\delta_{\tilde{k},0} \delta_{\tilde{p},0} q_{k,p,n}^+ - \delta_{\alpha_3+\beta_3,0}) \\ &= \delta_{\tilde{k},0} \delta_{\tilde{p},0} \delta_{\alpha_2+\beta_2,\alpha_3+\beta_3} \delta_{\alpha_1,-\beta_1} w_1(\beta_1, \alpha_3 + \beta_3). \end{aligned}$$

Lemma 4.12 yields the result.  $\square$

With notations of Lemma 4.12, it is direct to check that for given  $\bar{\alpha} = (\alpha, \alpha', \dots, \alpha^{(n-1)})$  and  $\bar{\beta} = (\beta, \beta', \dots, \beta^{(n-1)})$ ,

$$r((M_{\bar{\beta}}^{\bar{\alpha}})^{\circ}) = \sum_{K,P} \delta_{h_{K,P},0} v_{K,P} \pi_+(t_{K,P}) \otimes \pi_-(u_{K,P}) \quad (47)$$

where  $K = (k, k', \dots, k^{(n-1)})$ ,  $P = (p, p', \dots, p^{(n-1)})$  with  $0 \leq k_i^{(j)} \leq |\alpha_i^{(j)}|$ ,  $0 \leq p_i^{(j)} \leq |\beta_i^{(j)}|$ ,

$$\begin{aligned} t_{K,P} &:= t_{k,p} t_{k',p'} \cdots t_{k^{(n-1)},p^{(n-1)}} , & u_{K,P} &:= u_{k,p} u_{k',p'} \cdots u_{k^{(n-1)},p^{(n-1)}} , \\ v_{K,P} &:= v_{k,p} v_{k',p'} \cdots v_{k^{(n-1)},p^{(n-1)}} , & h_{K,P} &:= h_{k,p} + h_{k',p'} + \cdots h_{k^{(n-1)},p^{(n-1)}} . \end{aligned}$$

In the following, we will use the shortcuts  $A_i := \alpha_i + \alpha'_i + \cdots + \alpha_i^{(n-1)}$ ,  $B_i := \beta_i + \beta'_i + \cdots + \beta_i^{(n-1)}$ .

In the case  $n = 2$ , we also note  $r_{K,P}^{\pm} := r_{k,p}^{\pm} + r_{k',p'}^{\pm}$  and  $q_{K,P,n}^{\pm} := q_{k',p',n}^{\pm} q_{k,p,n+r_{k',p'}^{\pm}}^{\pm}$ .

Thus, we have  $\pi_+(t_{K,P})\varepsilon_m = q_{K,P,m}^+ \varepsilon_{m+r_{K,P}^+}$  and  $\pi_-(u_{K,P})\varepsilon_m = q_{K,P,n}^- \varepsilon_{m+r_{K,P}^-}$ .

We also introduce, still for  $n = 2$ ,

$$\begin{aligned} v_{\beta_1,\alpha'_1,\beta'_1}(l,j) &:= \sum_{n=0}^{\infty} (q^{l+2nj} q_{n+\beta'_1+\alpha'_1+\beta_1,|\beta'_1+\alpha'_1+\beta_1|}^{\uparrow_{-\beta'_1-\alpha'_1-\beta_1}} q_{n+\beta'_1+\alpha'_1,|\beta_1|}^{\uparrow_{\beta_1}} q_{n+\beta'_1,|\alpha'_1|}^{\uparrow_{\alpha'_1}} q_{n,|\beta'_1|}^{\uparrow_{\beta'_1}} - \delta_{j,0}), \\ V_{\bar{\beta}}^{\bar{\alpha}} &:= 2[\beta_1\beta'_1 + (\beta_2 - \beta_3)(\beta'_2 - \beta'_3)] q^{2\beta_1(\alpha_2+\alpha_3)+2\beta'_1(\alpha'_2+\alpha'_3)} \\ &\quad \times v_{\beta_1,\alpha'_1,\beta'_1}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha'_1 + \beta'_1), A_3 + B_3). \end{aligned}$$

**Lemma 4.14.** *We have*

- (i)  $(\tau_1 \otimes \tau_1)(r(M_{\bar{\beta}}^{\alpha} M_{\bar{\beta}'}^{\alpha'})^{\circ}) = \beta_1 \beta'_1 \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$
- (ii)  $(\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r(M_{\bar{\beta}}^{\alpha} M_{\bar{\beta}'}^{\alpha'})^{\circ}) = \delta_{A_2+B_2, A_3+B_3} \delta_{A_1, -B_1} V_{\bar{\beta}}^{\bar{\alpha}}.$

- (iii)  $(\tau_1 \otimes \tau_1) (r(M_\beta^\alpha M_{\beta'}^{\alpha'} M_{\beta''}^{\alpha''})^0) = \beta_1 \beta_1' \beta_1'' \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$   
(iv)  $(\tau_1 \otimes \tau_1) (r(\delta(M_\beta^\alpha) M_{\beta'}^{\alpha'})^0) = -(\alpha_1' + \beta_1') \beta_1 \beta_1' \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}.$   
(v) In particular, if  $A$  is a  $\delta$ -one-form,

$$\begin{aligned} \oint A^2 |\mathcal{D}|^{-3} &= 2\beta_1 \beta_1' B_a(A^2)_{\bar{\alpha}}^{\bar{\beta}}, \\ \oint A^2 |\mathcal{D}|^{-2} &= 2V_{\bar{\beta}}^{\bar{\alpha}} B(A^2)_{\bar{\alpha}}^{\bar{\beta}}, \\ \oint A^3 |\mathcal{D}|^{-3} &= 2\beta_1 \beta_1' \beta_1'' B_a(A^3)_{\bar{\alpha}}^{\bar{\beta}}, \\ \oint \delta(A) A |\mathcal{D}|^{-3} &= \oint A \delta(A) |\mathcal{D}|^{-3} = 0. \end{aligned}$$

*Proof.* We have

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}, \quad (48)$$

$$\tau_1(\pi_-(u_{K,P})) = \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}. \quad (49)$$

and

$$\tau_0(\pi_+(t_{K,P})) = \delta_{r_{K,P},0}^+ \sum_{n=0}^{\infty} (q_{K,P,n}^+ - \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}), \quad (50)$$

$$\tau_0(\pi_-(u_{K,P})) = \delta_{r_{K,P},0}^- \sum_{n=0}^{\infty} (q_{K,P,n}^- - \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}). \quad (51)$$

(i) Equations (48) and (49) give  $(\tau_1 \otimes \tau_1) r(\underline{A} \underline{A}')^0 = \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} \lambda_{0,0}$ . A computation of  $v_{0,0}$  with  $\delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} = 1$  gives the result.

(ii) Equations (48) and (51) yield

$$\begin{aligned} \tau_1(\pi_+(t_{K,P})) \tau_0(\pi_-(u_{K,P})) &= \delta_{K,0} \delta_{P,0} \delta_{A_2+B_2, A_3+B_3} \delta_{A_1, -B_1} \\ &\quad \times v_{\beta_1, \alpha_1', \beta_1'}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha_1' + \beta_1'), A_3 + B_3). \end{aligned}$$

Equations (50) and (49) yield

$$\begin{aligned} \tau_0(\pi_+(t_{K,P})) \tau_1(\pi_-(u_{K,P})) &= \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_2+B_2, A_3+B_3} \delta_{A_1, -B_1} \\ &\quad \times v_{\beta_1, \alpha_1', \beta_1'}((\alpha_2 + \beta_2 + \alpha_3 + \beta_3)(\alpha_1' + \beta_1'), A_3 + B_3) \end{aligned}$$

and the result follows.

(iii) With (47) a direct computation gives

$$\tau_1(\pi_+(t_{K,P})) = \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}, \quad (52)$$

$$\tau_1(\pi_-(u_{K,P})) = \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}. \quad (53)$$

Using (52) and (53),  $(\tau_1 \otimes \tau_1) (r(\underline{A} \underline{A}' \underline{A}'')^0) = \delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} v_{0,0}$ . A computation of  $v_{0,0}$  with  $\delta_{A_1, -B_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} = 1$  gives the result.

(iv) We have  $\delta(M_\beta^\alpha) M_{\beta'}^{\alpha'} = \delta(x) \delta(y) x' \delta(y') + x \delta^2(y) x' \delta(y')$  where  $x, x', y, y'$  are monomials ( $\pi$  omitted). Since

$$\pi(x) = \sum_k \binom{\alpha}{k} a_{+, \alpha_1}^{\hat{k}_1} a_{-, \alpha_1}^{k_1} b_{+}^{\hat{k}_2} b_{-}^{k_2} b_{+}^{* \hat{k}_3} b_{-}^{* k_3} =: \sum_k \binom{\alpha}{k} c_k,$$

we get  $\delta(\pi(x)) = \sum_k g(k) \binom{\alpha}{k} c_k$ .

Similarly,  $\delta(\pi(y)) = \sum_p g(p) \binom{\beta}{p} c_p$  and  $\delta^2(\pi(y)) = \sum_p g(p)^2 \binom{\beta}{p} c_p$ .

Thus, with  $c_{K,P} := c_k c_p c_{k'} c_{p'}$ ,

$$\delta(x)\delta(y)x'\delta(y') = \sum_{K,P} g(k)g(p)g(p') \binom{\alpha}{K} \binom{\beta}{P} c_{K,P},$$

$$x\delta^2(y)x'\delta(y') = \sum_{K,P} g(p)^2 g(p') \binom{\alpha}{K} \binom{\beta}{P} c_{K,P},$$

$$r(\delta(M_\beta^\alpha)M_{\beta'}^{\alpha'})^0 = \sum_{K,P} \delta_{h_{K,P},0} (g(k) + g(p)) g(p) g(p') \binom{\alpha}{K} \binom{\beta}{P} r(c_{K,P}) =: \sum_{K,P} \lambda_{K,P} r(c_{K,P}).$$

Since  $r(c_k) = (-q)^{k_1} (-1)^{\alpha_2+\alpha_3} \pi_+(t_k) \otimes \pi_-(u_k)$  with  $t_k, u_k$  defined by

$$t_k := a_{\alpha_1}^{\widehat{k}_1} b^{k_1} a^{\widehat{k}_2} b^{k_2} a^{*\widehat{k}_3} b^{k_3} \text{ and } u_k := a_{\alpha_1}^{\widehat{k}_1} b^{k_1} b^{\widehat{k}_2} a^{*k_2} b^{\widehat{k}_3} a^{k_3},$$

we get

$$r(\delta(M_\beta^\alpha)M_{\beta'}^{\alpha'})^0 = \sum_{K,P} \lambda_{K,P} (-q)^{k_1+k'_1+p_1+p'_1} (-1)^{A_2+A_3+B_2+B_3} \pi_+(t_{K,P}) \otimes \pi_-(u_{K,P})$$

where  $t_{K,P} = t_k t_p t_{k'} t_{p'}$  and  $u_{K,P} = u_k u_p u_{k'} u_{p'}$ . Direct computations yield

$$\begin{aligned} \tau_1(\pi_+(t_{K,P})) &= \delta_{K,0} \delta_{P,0} \delta_{A_1+A_2-A_3+B_1+B_2-B_3,0}, \\ \tau_1(\pi_-(u_{K,P})) &= \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{A_1-A_2+A_3+B_1-B_2+B_3,0}. \end{aligned}$$

The result follows.

(v) For the last equality, note that by (iv)

$$\oint \delta(A) A |\mathcal{D}|^{-3} = -2 \sum_{\alpha_1, \alpha'_1, \beta_1, \beta'_1} (\alpha'_1 + \beta'_1) \beta_1 \beta'_1 A_{\alpha_1 000}^{\beta_1 00} A_{\alpha'_1 00}^{\beta'_1 00} \delta_{\alpha_1 + \alpha'_1 + \beta_1 + \beta'_1, 0}.$$

The following change of variables  $\alpha_1 \leftrightarrow \alpha'_1$ ,  $\beta_1 \leftrightarrow \beta'_1$ , implies by symmetry that this is equal to zero.  $\square$

For a given  $\delta$ -1-form  $A$ , we say that  $A$  is homogeneous of degree in  $a$  equal to  $n \in \mathbb{Z}$  if it is a linear combination of  $M_\beta^\alpha$  such that  $\alpha_1 + \beta_1 = n$ . From Lemma 4.14 (iv) we get,

**Corollary 4.15.** *Let  $A, A'$  be two  $\delta$ -1-forms, then*

$$\begin{aligned} \oint (A |\mathcal{D}|^{-1})^2 &= \oint A^2 |\mathcal{D}|^{-2}, \\ \oint A |\mathcal{D}|^{-1} A' |\mathcal{D}|^{-1} &= \oint A A' |\mathcal{D}|^{-2} - n \oint A A' |\mathcal{D}|^{-3}, \text{ when } A' \text{ homogenous of degree } n. \end{aligned}$$

**Lemma 4.16.** *We have*

$$\begin{aligned} (i) \quad (\tau_1 \otimes \tau_1) \tilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) &= \beta_1 \beta'_1 \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0}, \\ (ii) \quad (\tau_0 \otimes \tau_1 + \tau_1 \otimes \tau_0) \tilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) &= \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} (\beta'_1 w_\beta^\alpha \delta_{\alpha'_2 + \beta'_2 + \alpha'_3 + \beta'_3, 0} \delta_{\alpha_2 + \beta_2, \alpha_3 + \beta_3} \\ &\quad + \beta_1 w_{\beta'}^{\alpha'} \delta_{\alpha_2 + \beta_2 + \alpha_3 + \beta_3, 0} \delta_{\alpha'_2 + \beta'_2, \alpha'_3 + \beta'_3}). \end{aligned}$$

- (iii)  $(\tau_1 \otimes \tau_1) \tilde{\rho}(M_\beta^\alpha M_{\beta'}^{\alpha'} J M_{\beta''}^{\alpha''} J^{-1}) = \beta_1 \beta_1' \beta_1'' \delta_{\alpha_1 + \alpha_1', -\beta_1 - \beta_1'} \delta_{\alpha_1'', -\beta_1''} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} \cdot$   
(iv)  $(\tau_1 \otimes \tau_1) \tilde{\rho}(\delta(M_\beta^\alpha) J M_{\beta'}^{\alpha'} J^{-1}) = -(\alpha_1' + \beta_1') \beta_1 \beta_1' \delta_{\alpha_1, -\beta_1} \delta_{\alpha_1', -\beta_1'} \delta_{A_2, 0} \delta_{A_3, 0} \delta_{B_2, 0} \delta_{B_3, 0} \cdot$   
(v) In particular, if  $A$  and  $A'$  are  $\delta$ -one forms,

$$\begin{aligned} \oint A J A' J^{-1} |\mathcal{D}|^{-3} &= 2(\beta_1 A_{-\beta_1 00}^{\beta_1 00})(\beta_1 \bar{A}'_{-\beta_1 00}^{\beta_1 00}), \\ \oint A J A' J^{-1} |\mathcal{D}|^{-2} &= 2(\beta_1 \bar{A}'_{-\beta_1 00}^{\beta_1 00})(w_\beta^\alpha B(A)_\alpha^\beta) + 2(\beta_1 A_{-\beta_1 00}^{\beta_1 00})(w_\beta^\alpha B(\bar{A}')_\alpha^\beta), \\ \oint A^2 J A' J^{-1} |\mathcal{D}|^{-3} &= 2(\beta_1 \bar{A}'_{-\beta_1 00}^{\beta_1 00})(\beta_1 \beta_1' B_a(A^2)_{\bar{\alpha}}^{\bar{\beta}}), \\ \oint \delta(A) J A J^{-1} &= 0. \end{aligned}$$

*Proof.* (i) Following notations of Lemma 4.12, we have

$$M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1} = \sum_{K, P} v_{K, P} c_{k, p} J c_{k', p'} J^{-1}$$

where  $K = (k, k')$ ,  $P = (p, p')$ ,  $\lambda_{K, P} = g(p)g(p')v_k v_{k'} w_p w_{p'}$ . Thus,

$$\tilde{\rho}(M_\beta^\alpha J M_{\beta'}^{\alpha'} J^{-1}) = (-1)^{A_2 + A_3 + B_2 + B_3} \sum_{K, P} (-q)^{k_1 + k_1' + p_1 + p_1'} \lambda_{K, P} T_{K, P}^+ \otimes T_{K, P}^-$$

where  $T_{K, P}^+ := \pi'_+(t_k t_p) \widehat{\pi}_+(t_{k'} t_{p'})$  and  $T_{K, P}^- := \pi'_-(u_k u_p) \widehat{\pi}_-(u_{k'} u_{p'})$  with

$$\begin{aligned} t_k &:= a_{\alpha_1}^{\widehat{k}_1} b_{\alpha_1}^{*k_1} a_{\alpha_1}^{\widehat{k}_2} b_{\alpha_1}^{*k_2} a_{\alpha_1}^{*\widehat{k}_3} b_{\alpha_1}^{k_3}, \\ u_k &:= a_{\alpha_1}^{\widehat{k}_1} b_{\alpha_1}^{*k_1} b_{\alpha_1}^{\widehat{k}_2} a_{\alpha_1}^{*k_2} b_{\alpha_1}^{*\widehat{k}_3} a_{\alpha_1}^{k_3}. \end{aligned}$$

A direct computation leads to

$$\begin{aligned} \tau_1(T_{K, P}^+) &= \delta_{K, 0} \delta_{P, 0} \delta_{\alpha_1 + \alpha_2 - \alpha_3 + \beta_1 + \beta_2 - \beta_3, 0} \delta_{\alpha_1' + \alpha_2' - \alpha_3' + \beta_1' + \beta_2' - \beta_3', 0}, \\ \tau_1(T_{K, P}^-) &= \delta_{\tilde{K}, 0} \delta_{\tilde{P}, 0} \delta_{\alpha_1 - \alpha_2 + \alpha_3 + \beta_1 - \beta_2 + \beta_3, 0} \delta_{\alpha_1' - \alpha_2' + \alpha_3' + \beta_1' - \beta_2' + \beta_3', 0} \end{aligned}$$

which gives the result.

(ii) Using the commutation relations on  $\mathcal{A}$ , we see that there are real functions of  $(K, P)$ , noted  $\sigma_{K, P}^t$  and  $\sigma_{K, P}^u$  such that

$$\begin{aligned} T_{K, P}^+ &= q^{\sigma_{K, P}^t} \pi'_+(t_{k, p}) \widehat{\pi}_+(t_{k', p'}), \\ T_{K, P}^- &= q^{\sigma_{K, P}^u} \pi'_-(u_{k, p}) \widehat{\pi}_-(u_{k', p'}), \\ t_{k, p} &:= a_{\alpha_1}^{\widehat{k}_1} a_{\alpha_1}^{\widehat{k}_2} a_{\alpha_1}^{*\widehat{k}_3} a_{\beta_1}^{\widehat{p}_1} a_{\beta_1}^{\widehat{p}_2} a_{\beta_1}^{*\widehat{p}_3} b_{\alpha_1}^{*k_1} b_{\beta_1}^{*p_1} b_{\beta_1}^{*k_2 + p_2} b_{\beta_1}^{k_3 + p_3}, \\ u_{k, p} &:= a_{\alpha_1}^{\widehat{k}_1} a_{\alpha_1}^{*k_2} a_{\alpha_1}^{k_3} a_{\beta_1}^{\widehat{p}_1} a_{\beta_1}^{*p_2} a_{\beta_1}^{p_3} b_{\alpha_1}^{*k_1} b_{\beta_1}^{*p_1} b_{\beta_1}^{\widehat{k}_2 + \widehat{p}_2} b_{\beta_1}^{*\widehat{k}_3 + \widehat{p}_3}. \end{aligned}$$

We have, under the hypothesis  $\tau_1(T_{K, P}^-) = 1$ ,

$$\begin{aligned} \widehat{\pi}_+(t_{k', p'}) \varepsilon_{m, 2j} &= (-1)^{\lambda'} q^{(2j-m)\lambda'} q_{2j-m-s+\beta_1', |\alpha_1'|}^{\uparrow \alpha_1'} q_{2j-m-s, |\beta_1'|}^{\uparrow \beta_1'} \varepsilon_{m+s, 2j}, \\ s &:= -\alpha_2' + \alpha_3' - \beta_2' + \beta_3' = \alpha_1' + \beta_1', \\ \lambda' &:= \alpha_2' + \alpha_3' + \beta_2' + \beta_3', \\ \lambda &:= \alpha_2 + \alpha_3 + \beta_2 + \beta_3 \\ \tau_1(T_{K, P}^+) &= \delta_{\lambda, 0} \delta_{\lambda', 0}. \end{aligned}$$

and then,

$$\begin{aligned}(T_{K,P}^+)_{m,2j} &= q^{\sigma_{K,P}^t + s\lambda} (-1)^{\lambda'} q^{(2j-m)\lambda' + m\lambda} F_m F'_{2j-m} \delta_{A_1+B_1,0}, \\ F'_{2j-m} &:= q_{2j-m-\alpha'_1, |\alpha'_1|}^{\uparrow_{\alpha'_1}} q_{2j-m-\alpha'_1-\beta'_1, |\beta'_1|}^{\uparrow_{\beta'_1}}, \\ F_m &:= q_{m-\alpha_1, |\alpha_1|}^{\uparrow_{\alpha_1}} q_{m-\beta_1-\alpha_1, |\beta_1|}^{\uparrow_{\beta_1}}.\end{aligned}$$

Following the proof of Lemma 4.7, we see that  $\tau_0(T_{K,P}^+)$  is possibly nonzero only in the two cases  $\lambda' = 0$  or  $\lambda = 0$ .

Suppose first  $\lambda = \lambda' = 0$ . In that case, we have

$$\tau_0(T_{K,P}^+) = \lim_{2j \rightarrow \infty} \sum_{m=0}^{2j} ((q_{m, |\beta_1|}^{\uparrow_{\beta_1}} q_{2j-m, |\beta'_1|}^{\uparrow_{\beta'_1}})^2 - 1) = \sum_{m=0}^{\infty} ((q_{m, |\beta_1|}^{\uparrow_{\beta_1}})^2 - 1) + \sum_{m=0}^{\infty} ((q_{m, |\beta'_1|}^{\uparrow_{\beta'_1}})^2 - 1)$$

where the second equality comes from Lemma 4.17.

In the case  $(\lambda = 0, \lambda' > 0)$ , we get  $\alpha'_1 = -\beta'_1$  and thus,

$$(T_{K,P}^+)_{m,2j} = q^{\sigma_{K,P}^t} q^{m\lambda} (q_{m, |\beta_1|}^{\uparrow_{\beta_1}} q_{2j-m, |\beta'_1|}^{\uparrow_{\beta'_1}})^2 \delta_{\alpha_1+\beta_1, 0}.$$

Let us note  $U_{2j} = \sum_{m=0}^{2j} q^{m\lambda} (q_{m, |\beta_1|}^{\uparrow_{\beta_1}} q_{2j-m, |\beta'_1|}^{\uparrow_{\beta'_1}})^2$  and  $L_{2j} = \sum_{m=0}^{2j} q^{m\lambda} (q_{m, |\beta_1|}^{\uparrow_{\beta_1}})^2$ .

Suppose  $\beta'_1 > 0$ . Since  $(q_{2j-m, |\beta'_1|}^{\uparrow_{\beta'_1}})^2 - 1 = \sum_{|p|_1 \neq 0, p_i \in \{0,1\}} (-1)^{|p|_1} q^{r_p} q^{2(2j-m)|p|_1}$  where we have  $r_p = 2 + \dots + 2\beta'_1$ . As in the proof of Lemma 4.7 (ii), we can conclude that  $U_{2j} - L_{2j}$  converges to 0. The case  $\beta'_1 \leq 0$  is similar.

In the other case  $(\lambda > 0, \lambda' = 0)$ , the arguments are the same, replacing  $\lambda$  by  $\lambda'$  and  $\alpha_1, \beta_1$  by  $\alpha'_1, \beta'_1$ . Finally,

$$\begin{aligned}\tau_0(T_{K,P}^+) \tau_1(T_{K,P}^-) &= \delta_{\tilde{K},0} \delta_{\tilde{P},0} \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} (\delta_{\lambda',0} \delta_{\alpha_2+\beta_2, \alpha_3+\beta_3} s_{\alpha, \beta} + \delta_{\lambda,0} \delta_{\alpha'_2+\beta'_2, \alpha'_3+\beta'_3} s_{\alpha', \beta'}), \\ s_{\alpha\beta} &:= q^{\beta_1(\alpha_3-\alpha_2)} \sum_{m=0}^{\infty} (q^{m\lambda} (q_{m, |\beta_1|}^{\uparrow_{\beta_1}})^2 - \delta_{\lambda,0}).\end{aligned}$$

A similar computation of  $\tau_0(T_{K,P}^-)$  can be done following the same arguments. We find eventually

$$\tau_1(T_{K,P}^+) \tau_0(T_{K,P}^-) = \delta_{K,0} \delta_{P,0} \delta_{\alpha_1, -\beta_1} \delta_{\alpha'_1, -\beta'_1} (\delta_{\lambda',0} \delta_{\alpha_2+\beta_2, \alpha_3+\beta_3} s_{\alpha, \beta} + \delta_{\lambda,0} \delta_{\alpha'_2+\beta'_2, \alpha'_3+\beta'_3} s_{\alpha', \beta'})$$

and the result follows.

(iii) The same arguments of (i) apply here with minor changes.

(iv) follows from a slight modification of the proof of Lemma 4.14 (iv).

(v) is a straightforward consequence of (i, ii, iii, iv).  $\square$

**Lemma 4.17.** *Let  $\beta, \beta' \in \mathbb{Z}$ . Then,*

$$\lim_{2j \rightarrow \infty} \sum_{m=0}^{2j} ((q_{m, |\beta|}^{\uparrow_{\beta}} q_{2j-m, |\beta'|}^{\uparrow_{\beta'}})^2 - 1) = \sum_{m=0}^{\infty} ((q_{m, |\beta|}^{\uparrow_{\beta}})^2 - 1) + \sum_{m=0}^{\infty} ((q_{m, |\beta'|}^{\uparrow_{\beta'}})^2 - 1).$$

*Proof.* We give a proof for  $\beta$  and  $\beta' > 0$ , the other cases being similar.

Since  $(q_{m,|\beta|}^{\uparrow\beta})^2 = \sum_{p_i \in \{0,1\}} (-1)^{|p|_1} q^{r_p} q^{2|p|_1 m}$  where  $p = (p_1, \dots, p_\beta)$  and  $r_p := 2(p_1 + \dots + \beta p_\beta)$ , we get, with the notations  $\lambda_{p,p'} := (-1)^{|p+p'|_1} q^{r_p+r_{p'}}$  and  $U_{2j} := \sum_{m=0}^{2j} (q_{m,|\beta|}^{\uparrow\beta} q_{2j-m,|\beta'|}^{\uparrow\beta'})^2 - 1$ ,

$$\begin{aligned} U_{2j} &= \sum_{m=0}^{2j} \sum_{|p+p'|_1 > 0} \lambda_{p,p'} q^{2|p|_1 m + 2|p'|_1 (2j-m)} \\ &= \sum_{|p|_1 \geq |p'|_1, |p|_1 > 0} \lambda_{p,p'} V_{2j,p,p'} + \sum_{|p|_1 < |p'|_1, |p'|_1 > 0} \lambda_{p,p'} V'_{2j,p,p'} \end{aligned}$$

where

$$V_{2j,p,p'} = q^{4j|p'|_1} \sum_{m=0}^{2j} q^{2(|p|_1 - |p'|_1)m}, \quad V'_{2j,p,p'} = q^{4j|p|_1} \sum_{m=0}^{2j} q^{2(|p'|_1 - |p|_1)m}.$$

It is clear that  $V_{2j,p,p'}$  has 0 for limit when  $j \rightarrow \infty$  when  $|p'|_1 > 0$ , and  $V'_{2j,p,p'}$  has 0 for limit when  $j \rightarrow \infty$  when  $|p|_1 > 0$ . As a consequence,

$$U_{2j} = \sum_{|p|_1 > 0} \lambda_{p,0} V_{2j,p,0} + \sum_{|p'|_1 > 0} \lambda_{0,p'} V'_{2j,0,p'} + o(1).$$

The result follows as

$$\sum_{m=0}^{2j} ((q_{m,|\beta|}^{\uparrow\beta})^2 - 1) = \sum_{|p|_1 > 0} \lambda_{p,0} V_{2j,p,0} \text{ and } \sum_{m=0}^{2j} ((q_{m,|\beta'|}^{\uparrow\beta'})^2 - 1) = \sum_{|p'|_1 > 0} \lambda_{0,p'} V'_{2j,0,p'}. \quad \square$$

*Proof of Theorem 4.11.* The result follows from Lemmas 4.13, 4.14 (v) and 4.16 (v).  $\square$

## 4.7 Proof of Theorem 4.3 and corollaries

**Lemma 4.18.** *We have on  $SU_q(2)$ ,*

- (i)  $f|\mathcal{D}|^{-3} = 2.$
- (ii)  $f|\mathcal{D}|^{-2} = 0.$
- (iii)  $f|\mathcal{D}|^{-1} = -\frac{1}{2}.$
- (iv)  $\zeta_{\mathcal{D}}(0) = 0.$

*Proof.* (iv) We have by definition

$$\zeta_{\mathcal{D}}(s) := \text{Tr}(|\mathcal{D}|^{-s}) = \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle v_{m,l}^j, |\mathcal{D}|^{-s} v_{m,l}^j \rangle.$$

Since  $|\mathcal{D}|^{-s} v_{m,l}^j = \begin{pmatrix} d_{j+}^{-s} & 0 \\ 0 & d_j^{-s} \end{pmatrix} v_{m,l}^j$  where  $d_j := 2j + \frac{1}{2}$ , we get

$$\zeta_{\mathcal{D}}(s) = \sum_{2j=0}^{\infty} (2j+1)(2j+2) d_{j+}^{-s} + \sum_{2j=1}^{\infty} (2j+1)(2j) d_j^{-s} = 2 \sum_{2j=0}^{\infty} (2j+1)(2j) d_j^{-s}.$$

With the equalities  $(2j+1)(2j) = d_j^2 - \frac{1}{4}$  and  $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$  (here  $\zeta(s, x) := \sum_{n \in \mathbb{N}} \frac{1}{(n+x)^s}$  is the Hurwitz zeta function and  $\zeta(s) := \zeta(s, 1)$  is the Riemann zeta function) we get

$$\zeta_{\mathcal{D}}(s) = 2(2^{s-2} - 1)\zeta(s-2) - \frac{1}{2}(2^s - 1)\zeta(s) \quad (54)$$

which entails that  $\zeta_{\mathcal{D}}(0) = 0$ .

(i, ii, iii) are direct consequences of equation (54).  $\square$

*Proof of Theorem 4.3.* It is a consequence of Lemma 4.18 and Theorems 4.1, 4.11.  $\square$

As we have seen, the computation of noncommutative integral on  $SU_q(2)$  leads to certain function of  $A$  which filter some symmetry on the degree in  $a, a^*, b, b^*$  of the canonical decomposition. Precisely, it is the balanced features that appear and the following functions of  $A^n, n \in \{1, 2, 3\}$ :

$$\oint A^n |\mathcal{D}|^{-p} \quad (55)$$

where  $1 \leq n \leq p \leq 3$ . We will see in the next section a method for the computation of these integrals.

**Corollary 4.19.** *Let  $u$  be a unitary in  $C^\infty(SU_q(2))$  and  $\gamma_u(\mathbb{A}) := \pi(u)\mathbb{A}\pi(u^*) + \pi(u)d\pi(u^*)$  be a gauge-variant of  $\mathbb{A}$ . Then the following term of Theorem 4.3 are gauge invariant*

$$\oint A |\mathcal{D}|^{-3}, \quad \oint A^2 |\mathcal{D}|^{-3} - \oint A |\mathcal{D}|^{-2}, \quad -2 \oint A |\mathcal{D}|^{-1} + \oint A^2 |\mathcal{D}|^{-2} - \frac{2}{3} \oint A^3 |\mathcal{D}|^{-3}.$$

*Proof.* It is sufficient to remark that all terms  $\oint |D_{\mathbb{A}}|^{-k}$  and  $\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$  in spectral action (4) are gauge invariant. This can also be seen via the computation  $D_{\gamma_u(\mathbb{A})} = V_u \mathcal{D} V_u^* + V_u P_0 V_u^*$  where  $P_0$  is the projection on  $\text{Ker } \mathcal{D}$  and  $V_u = \pi(u)J\pi(u)J^{-1}$  and  $\oint |D_{\mathbb{A}}|^{n-k} = \text{Res}_{s=n-k} \text{Tr} (|D_{\mathbb{A}}|^{n-k})$  (see [22, Prop. 5.1 (iii) and Prop. 4.8].)  $\square$

**Corollary 4.20.** *In the case of the spectral action without the reality operator (i.e.  $D_{\mathbb{A}} = \mathcal{D} + \mathbb{A}$ ), we get*

$$\begin{aligned} \oint |D_{\mathbb{A}}|^{-3} &= 2, & \oint |D_{\mathbb{A}}|^{-2} &= -2 \oint A |\mathcal{D}|^{-3}, & \oint |D_{\mathbb{A}}|^{-1} &= -\frac{1}{2} - \oint A |\mathcal{D}|^{-2} + \oint A^2 |\mathcal{D}|^{-3}, \\ \zeta_{D_{\mathbb{A}}}(0) &= -\oint A |\mathcal{D}|^{-1} + \frac{1}{2} \oint A^2 |\mathcal{D}|^{-2} - \frac{1}{3} \oint A^3 |\mathcal{D}|^{-3}. \end{aligned}$$

As a consequence, if  $\mathbb{A}$  is a one-form such that  $\oint A |\mathcal{D}|^{-3} = 0$ , then the scale invariant term of the spectral action with or without  $J$  is exactly the same modulo a global factor of 2.

## 5 Differential calculus on $SU_q(2)$ and applications

### 5.1 The sign of $\mathcal{D}$

There are multiple differential calculi on  $SU_q(2)$ , see [33, 39]. Thanks to [36, Theorem 3], the  $3D$  and  $4D_{\pm}$  differential calculi do not coincide with the one considers here: the right multiplication of one-forms by an element in the algebra  $A$  is a consequence of the chosen Dirac operator which was introduced according to some equivariance properties with respect to the duality between the two Hopf algebras  $SU_q(2)$  and  $\mathcal{U}_q(su(2))$ .

It is known that the Fredholm module associated to  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  is one-summable since  $[F, \pi(x)]$  is trace-class for all  $x \in \mathcal{A}$ . In fact, more can be said about  $F$ <sup>1</sup>:

**Proposition 5.1.** *Since*

$$F = \frac{1}{1-q^2} (\underline{\pi}(a^*) d\underline{\pi}(a) + q^2 \underline{\pi}(b) d\underline{\pi}(b^*) + q^2 \underline{\pi}(a) d\underline{\pi}(a^*) + q^2 \underline{\pi}(b^*) d\underline{\pi}(b)), \quad (56)$$

*$F$  is a central one-form modulo  $OP^{-\infty}$ .*

---

<sup>1</sup>Note that a similar result for a different spectral triple over  $SU_q(2)$  when  $q = 0$  was obtained in [13, eq. (48)]

*Proof.* Forgetting  $\underline{\pi}$ , this follows from

$$\begin{aligned}
a^* \delta a + q^2 b \delta b^* + q^2 a \delta a^* + q^2 b^* \delta b \\
&= (a_+^* + a_-^*)(a_+ - a_-) + q^2 (b_+ + b_-)(b_-^* - b_+^*) + q^2 (a_+ + a_-)(a_-^* - a_+^*) \\
&\quad + q^2 (b_+^* + b_-^*)(b_+ - b_-) \\
&= [a_+^* a_+ - q^2 a_+ a_+^* + q^2 b_+^* b_+ - q^2 b_+ b_+^*] + R = (1 - q^2) + R
\end{aligned} \tag{57}$$

by (16) where we check that the remainder  $R$  is zero:

$$\begin{aligned}
R = -[a_+^* a_- + q^2 b_+^* b_-] + [a_-^* a_+ + q^2 b_-^* b_+] - [a_-^* a_- - q^2 a_- a_-^* + q^2 b_-^* b_- - q^2 b_- b_-^*] \\
+ (q^2 a_+ a_-^* + q^2 q_-^* b_+) - (a_+^* a_- + q^2 b_+^* b_-),
\end{aligned}$$

thus, applying (19), (20), (21),  $R = +(q^2 a_+ a_-^* + q^2 q_-^* b_+) - (a_+^* a_- + q^2 b_+^* b_-) = 0$  using commutation relations (15).

Now, replacing  $\delta$  by  $d$  in (57) gives (56) since  $F$  commute with  $a_\pm$ ,  $b_\pm$  and  $F$  is central by (28).  $\square$

**Proposition 5.2.** *The one-form in (56) is in fact exactly a function of the Dirac operator  $D$ :*

$$\pi(a^*) d\pi(a) + q^2 \pi(b) d\pi(b^*) + q^2 \pi(a) d\pi(a^*) + q^2 \pi(b^*) d\pi(b) = \xi_q(\mathcal{D}) = F \xi_q(|\mathcal{D}|), \tag{58}$$

where  $\xi_q(s) := q^{\frac{[2s]-2s}{[s+1/2][s-1/2]}}$ .

Moreover,  $F = \lim_{q \rightarrow 0} \xi_q(D)$ .

*Proof.* First, let us observe that the one-form  $\omega$  in (58) is invariant under the action of the  $\mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2))$ :  $h \triangleright \omega = \epsilon(h) \omega$  for any  $h \in \mathcal{U}_q(su(2)) \times \mathcal{U}_q(su(2))$ . For instance, using notations of [21]

$$e \triangleright \omega = q^{\frac{1}{2}} a^* db + q^2 \left( -q^{\frac{1}{2}-1} b da^* + q^{-\frac{1}{2}} b da^* - q^{-1-\frac{1}{2}} a^* db \right) = 0 = \epsilon(e) \omega.$$

Therefore, since both the representation  $\pi$  as well as the operator  $D$  are equivariant, the image of  $\omega$  must be diagonal in the spinorial base. A tedious computation with the full spinorial representation  $\pi$  given in (10) yields

$$\begin{aligned}
\langle v_{ml}^{j\uparrow}, \omega v_{ml}^{j\uparrow} \rangle &= \frac{q^{8j+8} - q^{8j+6} - (4j+3) q^{4j+6} + (8j+6) q^{4j+4} - (4j+3) q^{4j+2} - q^2 + 1}{(q^{4j+4} - 1)(q^{4j+2} - 1)} = \xi_q(2j + \frac{3}{2}), \\
\langle v_{ml}^{j\downarrow}, \omega v_{ml}^{j\downarrow} \rangle &= \frac{-q^{8j+4} + q^{8j+2} + (4j+1) q^{4j+4} - (8j+2) q^{4j+2} + (4j+1) q^{4j} + q^2 - 1}{(q^{4j+2} - 1)(q^{4j} - 1)} = -\xi_q(2j + \frac{1}{2}).
\end{aligned}$$

These expressions have a clear  $q = 0$  limit equal respectively to 1 and -1, so  $\omega \rightarrow F$  as  $q \rightarrow 0$ .  $\square$

In the  $q = 1$  limit, these expressions yields identically 0, which is confirmed by the fact that all one-forms are central, it could be expressed as  $d(aa^* + bb^*) = d1$ .

Note that since the invariant one-form we constructed differs by  $OP^{-\infty}$  from  $F$ , hence any commutator with it will be itself in  $OP^{-\infty}$ .

We do not know if a central form  $\omega$  is automatically invariant by the action of both  $U_q(su(2))$ , that is:  $h \triangleright \omega = \epsilon(h) \omega$ .

**Proposition 5.3.** *The order one calculus up to  $OP^{-\infty}$  is not universal.*



*Proof.* Let us take the one-form  $\omega_F$  from (56), which gives  $F$ . Then, for any  $x \in \mathcal{A}(SU_q(2))$  we have  $\pi(x\omega_F - \omega_F x) = 0$ .  $\square$

**Corollary 5.4.** *Still modulo  $OP^{-\infty}$ ,  $1 \in \pi(\Omega_u^2(\mathcal{A}))$ .*

*Proof.*  $1 = F^2$  is by definition in  $\pi(\Omega_u^2(\mathcal{A}))$ .  $\square$

In fact, one checks, using (16), (19), (22) that

$$q^2 da da^* - da^* da = 1 - q^2 \quad (59)$$

showing again that  $1 \in \pi(\Omega_u^2(\mathcal{A}))$ .

Similarly, using (15) and (17), (22), (23), we get still up to  $OP^{-\infty}$

$$\begin{aligned} q da db &= db da, & q da db^* &= db^* da, \\ da^* db &= q db da^*, & da^* db^* &= q db^* da^* \\ db db^* &= db^* db, & da da^* + db db^* &= -1. \end{aligned} \quad (60)$$

The use of the last equality of (60) and (59) gives

**Proposition 5.5.** *Up to  $OP^{-\infty}$ ,  $F$  is not a (universal) closed one-form, as*

$$da^* da + q^2 da da^* + q^2 db^* db + q^2 db db^* = -1 - q^2. \quad (61)$$

## 5.2 The ideal $\mathcal{R}$

In order to perform explicit calculations of all terms of the spectral action, we observe that each  $\delta$ -one-form could be expressed in terms of  $x\delta(z)y$ , where  $z$  is one of the generators  $a, a^*, b, b^*$  and  $x, y$  are some elements of the algebra  $\mathcal{A}(SU_q(2))$ .

Then, for the computation of  $\oint x dz y |D|^{-1}$  we can use the trace property of the noncommutative integral to get:

$$\oint x \delta(z) y |D|^{-1} = \oint y x \delta(z) |D|^{-1} + \oint x \delta(z) |D|^{-1} \delta(y) |D|^{-1}.$$

Therefore, the problem of calculating the tadpole-like integral could be in effect reduced to the calculation of much simpler integrals:  $\oint x \delta(z)$  for all generators  $z$  and the integrals of higher order in  $|D|^{-1}$ .

However, it appears that the calculations of higher-order terms simplify a lot, when we further restrict the algebra by introducing an ideal, which is *invisible* to the parts of integral at dimension 2 and 3. For instance, consider the space of pseudodifferential operators  $T \in \Psi^0(\mathcal{A})$  of order less or equal to zero (see [16]), which satisfy

$$\oint T t |D|^{-2} = \oint t T |D|^{-2} = \oint T t |D|^{-3} = \oint t T |D|^{-3} = 0, \quad \forall t \in \Psi_0^0(\mathcal{A}). \quad (62)$$

The elements  $a_-, b_- b_+, b_- b_+^*$  and their adjoints are in this space up to  $OP^{-\infty}$ : this is due to the fact that in Theorem 3.4,  $\tau_1 \otimes \tau_1(r(x)) = 0$  when  $r(x) \in \pi_{\pm}(\mathcal{A}) \otimes \pi_{\pm}(\mathcal{A}) \bmod OP^{-\infty}$  contains tensor products of  $\pi_{\pm}(b)$  or  $\pi_{\pm}(b^*)$  since these elements are in the kernel of the grading  $\sigma$ .

**Definition 5.6.** Let  $R$  be the kernel in  $X$  of  $(\sigma \otimes \sigma) \circ r$  where  $r$  is the Hopf-map defined in (30) and  $\sigma$  is the symbol map and let  $\mathcal{R}$  be the vector space generated by  $R$  and  $RF$ .

Note that  $R$  is a  $*$ -ideal in  $X$  and

$$a_-, b_- b_+ (= q^2 b_+ b_-), b_- b_+^* \text{ are in } \mathcal{R}.$$

By construction and Theorem 3.4, any  $T \in \mathcal{R}$  satisfies (62) and  $\mathcal{R}$  is invariant by  $F$ .

Moreover, by (19),  $[b_-, b_-^*] \in R$ , so by (16) and (22),  $a_+^* a_+ - q^2 a_+ a_+^* - (1 - q^2) \in R$  and by (23),  $q a_+ b_- - b_- a_+ \in R$ .

It is interesting to quote, thanks to Theorem 3.4 that if  $x \in R$ , then  $\oint F x |D|^{-1} = 0$  while a priori,  $\oint x |D|^{-1} \neq 0$ .

Note that  $F \in \Psi^0(\mathcal{A})$  also satisfies (62) by Theorem 3.4 while  $F \notin \mathcal{R}$  since  $F^2 = 1$ .

Moreover other elements are in  $\mathcal{R}$  like for instance  $d(b^* b) = d(bb^*)$ :

$$\begin{aligned} \delta(bb^*) &= -\delta(aa^*) = -\delta a a^* - a \delta a^* = -(a_+ - a_-)(a_+^* + a_-^*) - (a_+ + a_-)(a_-^* - a_+^*) \\ &= 2(a_+ a_-^* - a_- a_+^*) \end{aligned}$$

is in  $R$  since  $a_- \in R$  yielding  $d(bb^*) \in RF$ .

We do not know if  $\mathcal{R}$  is equal to the subset of the algebra generated by  $\mathcal{B}$  and  $\mathcal{B}F$  satisfying (62).

**Lemma 5.7.**  $\mathcal{R}$  is a  $*$ -ideal in  $\Psi^0(\mathcal{A})$  which is invariant by  $F$ ,  $d$ ,  $\delta$ .

*Proof.* Since  $R$  is an ideal in  $X = \mathcal{B} \bmod OP^{-\infty}$  (see Remark 3.3),  $\mathcal{R}$  appears to be an ideal in  $\Psi^0(\mathcal{A}) \subset$  algebra generated by  $\mathcal{B}$  and  $\mathcal{B}F$ . Since  $\mathcal{R}$  is invariant by  $F$ , its invariance by  $d$  follows from its invariance by  $\delta$  which is true on the generators of  $R$ .  $\square$

Note that, according to Theorem 4.13,  $\oint da |D|^{-2} = \oint da |D|^{-3} = 0$  while  $\oint a^* da |D|^{-3} = 2$  which emphasize the role of "for all  $t$ " in (62).

**Lemma 5.8.** For any  $t \in \Psi_0^0(\mathcal{A})$  and  $T \in \mathcal{R}$ , we have  $\oint t T |D|^{-1} = \oint T t |D|^{-1}$ .

*Proof.* For any  $t \in \mathcal{B}$ , we have  $\oint T t |D|^{-1} = \oint t T |D|^{-1} + \oint T |D|^{-1} \delta(t) |D|^{-1}$  and moreover  $\oint T |D|^{-1} \delta(t) |D|^{-1} = \oint T \delta(t) |D|^{-2} - \oint T \delta^2(t) |D|^{-3}$  which comes from

$$\begin{aligned} |D|^{-1} \delta(t) |D|^{-1} &= \delta(t) |D|^{-2} + [|D|^{-1}, \delta(t)] |D|^{-1} = \delta(t) |D|^{-2} - |D|^{-1} \delta^2(t) |D|^{-2} \\ &= \delta(t) |D|^{-2} - \delta^2(t) |D|^{-3} + |D|^{-1} \delta^3(t) |D|^{-3}. \end{aligned}$$

So we get the result because  $T$  satisfies (62).  $\square$

**Lemma 5.9.** If  $\simeq$  means equality up to the ideal  $\mathcal{R}$ , the following rules with  $d(\cdot) = [\mathcal{D}, \cdot]$  of the first-order differential calculus hold (forgetting  $\pi$ )

$$\begin{array}{llll} a da \simeq da a, & a^* da \simeq -da^* a, & b da \simeq q da b, & b^* da \simeq q da b^*, \\ a da^* \simeq -da a^*, & a^* da^* \simeq da^* a^*, & b da^* \simeq q^{-1} da^* b, & b^* da^* \simeq q^{-1} da^* b^*, \\ a db \simeq q^{-1} db a, & a^* db \simeq q db a^*, & b db \simeq db b, & b^* db \simeq db b^* \simeq -b db^*, \\ a db^* \simeq q^{-1} db^* a, & a^* db^* \simeq q db^* a^*, & b db^* \simeq db^* b \simeq -b^* db, & b^* db^* \simeq db^* b^*. \end{array}$$

Moreover

$$a^* da - q^2 da a^* \simeq (1 - q^2) F, \quad q^2 a da^* - da^* a \simeq (1 - q^2) F. \quad (63)$$

*Proof.* The table follows from relations (7) and Lemma 3.2 with (28) (one can also use (15).) For instance, since  $a_- \in \mathcal{R}$ , using the fact that  $\mathcal{R}$  is invariant by  $F$ ,

$$\begin{aligned} b da &= (b_+ + b_-)(a_+ - a_-) F \simeq (b_+ + b_-)(a_+ + a_-) F = ba F = q ab F \simeq q(a_+ - a_-) F b \\ &= q da b \end{aligned}$$

or similarly,  $a^* da = (a_+^* + a_-^*)(a_+ - a_-) F \simeq (a_+^* - a_-^*)(a_+ + a_-) F = -da^* a$ .

The second of equivalence of (63) is just the adjoint of the first one that we prove now:

$$\begin{aligned} a^* da - q^2 da a^* &= (a_+^* + a_-^*)(a_+ - a_-) F - q^2 (a_+ - a_-) F (a_+^* + a_-^*) \\ &\simeq (a_+^* + a_-^*)(a_+ + a_-) F - q^2 (a_+ + a_-) (a_+^* + a_-^*) F = (a^* a - q^2 a a^*) F \\ &= (1 - q^2) F. \end{aligned} \quad \square$$

**Remark 5.10.** *The above written rules remain valid if  $dx$  is replaced by  $\delta(x)$  and  $F$  by 1.*

Working modulo  $\mathcal{R}$  simplifies the writing of a one-form:

**Lemma 5.11.** *(i) Every one-form  $A$  can be, up to elements from  $\mathcal{R}$ , presented as*

$$A \simeq x_a da + da^* x_{a^*} + x_b db + db^* x_{b^*},$$

where all  $x_*$  are the elements of  $\mathcal{A}$ .

*(ii) When  $A$  is selfadjoint,  $A$  can be written up to  $\mathcal{R}$  (not in a unique way, though) as*

$$A \simeq x_a da - da^* (x_a)^* + x_b db - db^* (x_b)^*,$$

where  $x_a, x_b$  are arbitrary elements of  $\mathcal{A}$ .

*Proof.* (i) A basis for one-forms consists of the following forms:  $a^\alpha b^\beta (b^*)^\gamma d(a^{\alpha'} b^{\beta'} (b^*)^{\gamma'})$ , where  $\alpha, \alpha' \in \mathbb{Z}$  and  $\beta, \gamma, \beta', \gamma' \in \mathbb{N}$ .

Using the Leibniz rule and the commutation rules within the algebra (up to the  $\mathcal{R}$  according to Lemma 5.9), we reduce the problem to the case of the forms:  $(a^\alpha b^\beta (b^*)^\gamma) dx (a^{\alpha'} b^{\beta'} (b^*)^{\gamma'})$ , where  $x$  can be either of the generators  $a, a^*, b, b^*$ . If  $x = b$  or  $x = b^*$ , the straightforward application of the rules of the differential calculus leads to the answer that the one-form could be expressed as:  $a^\alpha b^\beta (b^*)^\gamma db$  and  $db^* a^\alpha b^\beta (b^*)^\gamma$ .

Similar considerations for the case  $x = a, a^*$  lead to the remaining terms.

Note that the presentation is not unique, since there still might remain terms, which are in  $\mathcal{R}$ , for example:  $b^* db + db^* b = d(bb^*) \in \mathcal{R}$ .

(ii) is direct.  $\square$

Next we can start explicit calculation of the integrals, beginning with the tadpole terms.

Application of the Leibniz rule yields to a presentation of one-forms which is different from the one of previous lemma. Each  $\delta$ -one-form could be expressed, as a finite sum of the terms  $x\delta(z)y$ , where  $z$  is one of the generators  $a, a^*, b, b^*$  and  $x, y$  are some elements of the algebra  $\mathcal{A}(SU_q(2))$ .

**Proposition 5.12.** *For all  $x, y \in \mathcal{A}(SU_q(2))$  and  $z \in \{a, a^*, b, b^*\}$  we have*

$$\oint x\delta(z)y |D|^{-1} = \oint yx\delta(z) |D|^{-1} + \oint x\delta(z)\delta(y) |D|^{-2} - \oint x\delta(z)\delta^2(y) |D|^{-3}.$$

*Proof.* This is just the application of the trace property of the noncommutative integral, together with the identity:  $|D|^{-1}\delta(z)|D|^{-1} = -[|D|^{-1}, z]$ .  $\square$

**Remark 5.13.** *The computation of tadpole-like integrals is reduced to the integrals  $\oint x\delta(z)|\mathcal{D}|^{-1}$  for all generators  $z$  and the integrals of higher order in  $|\mathcal{D}|^{-2}$ . However, the calculations of higher-order terms simplify a lot after we use the relations which hold up to the ideal  $\mathcal{R}$ : this erases parts of integral depending on  $|\mathcal{D}|^{-2}$  and  $|\mathcal{D}|^{-3}$ . Thus, beside  $\oint x\delta(z)|\mathcal{D}|^{-1}$ , we only need to compute  $\oint x\delta(z)\delta(z')|\mathcal{D}|^{-2}$  where  $z$  and  $z'$  are generators, since all the  $|\mathcal{D}|^{-3}$  integrals have already been explicitly computed in section 4.6 (these integrals do not depend on  $q$ .) Besides the tadpole, the only integrals that need to be computed are  $\oint A|\mathcal{D}|^{-2}$  and  $\oint A^2|\mathcal{D}|^{-2}$  where  $A$  is a  $\delta$ -1-form. Working modulo  $\mathcal{R}$  and using again Leibniz rule, we only need to compute  $\oint x\delta(z)|\mathcal{D}|^{-2}$  and the previous integrals  $\oint x\delta(z)\delta(z')|\mathcal{D}|^{-2}$ .*

### 5.2.1 Operators $L_q$ and $M_q$

In the notation  $v_{l,m}^j$  of  $\mathcal{H}$ , we have already use the  $j$  dependence in (12) with  $J_q v_{m,l}^j := q^j v_{m,l}^j$ . Let  $L_q$  and  $M_q$  be the similar diagonal operators

$$\begin{aligned} L_q v_{m,l}^j &:= q^{2l} v_{m,l}^j, \\ M_q v_{m,l}^j &:= q^{2m} v_{m,l}^j. \end{aligned}$$

We immediately get

**Lemma 5.14.** *For  $n \in \mathbb{N}^*$ ,  $\text{f}(L_q)^n |\mathcal{D}^{-2}| = \text{f}(M_q)^n |\mathcal{D}^{-2}| = \frac{2}{1-q^{2n}}$ .*

*Proof.* We have

$$\begin{aligned} \text{Tr} (L_q^n |\mathcal{D}|^{-2-s}) &= \sum_{2j=0}^{\infty} \sum_{m=0}^{2j} \sum_{l=0}^{2j+1} \langle v_{m,l}^j, L_q^n |\mathcal{D}|^{-2-s} v_{m,l}^j \rangle \\ &= \sum_{2j=0}^{\infty} (2j+1) \frac{1-q^{2n(2j+2)}}{1-q^{2n}} d_{j+}^{-2-s} + \sum_{2j=0}^{\infty} (2j+1) \frac{1-q^{2n(2j+2)}}{1-q^{2n}} d_j^{-2-s} \\ &\sim_0 \frac{1}{1-q^{2n}} \left( \zeta(s+1, \frac{3}{2}) + \zeta(s+1, \frac{1}{2}) \right) \sim_e \frac{2}{1-q^{2n}} \zeta(s+1). \end{aligned}$$

where  $\sim_0$  means modulo a function holomorphic at 0. This gives the result for  $L_q^n$  and a similar computation can be done for  $M_q^n$ .  $\square$

The interest of these operators stems in

**Lemma 5.15.** *We have  $L_q M_q \in \mathcal{R}$ . Moreover,*

$$\begin{aligned} b\delta b^* &\simeq M_q - L_q, & b^*\delta b &\simeq L_q - M_q, & bb^* &\simeq L_q + M_q, \\ a\delta(a^*) &\simeq -aa^* \simeq L_q + M_q - 1, & a^*\delta a &\simeq a^*a \simeq 1 - q^2(L_q + M_q), \\ da da^* &\simeq L_q + M_q - 1, & da^* da &\simeq q^2(L_q + M_q) - 1, \\ b^{n-2}(b^*)^n db db &\simeq (L_q)^n + (M_q)^n, \\ b^{n-1}(b^*)^{n-1} db db^* &\simeq -(L_q)^n - (M_q)^n, \\ b^n(b^*)^{n-2} db^* db^* &\simeq (L_q)^n + (M_q)^n. \end{aligned}$$

*Proof.* Since  $L_q M_q = q^2 a_- a_-^* \in \mathcal{R}$ , we compute up to the ideal  $\mathcal{R}$

$$b \delta b^* = (b_+ + b_-)(b_-^* - b_+^*) \simeq -b_+ b_+^* + b_- b_-^* = M_q - L_q + L_q M_q(1 - q^2) \simeq M_q - L_q$$

and similarly for the other relations.  $\square$

## 5.2.2 Automorphisms of the algebra and symmetries of integrals

**Proposition 5.16.** *For any  $n \in \mathbb{N}^*$ ,*

$$\begin{aligned} \oint (bb^*)^n |\mathcal{D}|^{-1} &= \frac{-2(1+q^{2n})}{(1-q^{2n})^2}, \\ \oint (bb^*)^n b^* \delta b |\mathcal{D}|^{-1} &= \oint (bb^*)^n b \delta b^* |\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}, \\ \oint (bb^*)^n a da^* \mathcal{D}^{-1} &= \frac{-2q^{4n+2}-2q^{4n}-2q^{2n+2}+6q^{2n}}{(1-q^{2n})^2(1-q^{2n+2})}, \\ \oint (bb^*)^n a^* da \mathcal{D}^{-1} &= \frac{6q^{2n+2}-2q^{2n}-2q^2-2}{(1-q^{2n})^2(1-q^{2n+2})}. \end{aligned}$$

Note that the knowledge of these integral is enough for the computation of any term of the form  $\oint x \delta(z) |\mathcal{D}|^{-1}$ , where  $z$  is a generator, since any other  $\delta$ -one-form will be unbalanced.

To show this proposition, we will use few symmetries, properties of the ideal  $\mathcal{R}$  and replacement of  $\delta$ -one-forms in terms of  $L_q, M_q$  as above.

Let  $U$  be the following unitary operator on the Hilbert space:

$$U v_{m,l}^{j\uparrow} = (-1)^{m+l} v_{l,m}^{j\downarrow}, \quad U v_{m,l}^{j\downarrow} = (-1)^{m+l} v_{l,m}^{j\uparrow}.$$

Then, by explicit computations we have

$$U^* a U = a, \quad U^* a^* U = a^*, \quad U^* b U = b^*, \quad U^* b^* U = b, \quad \text{and} \quad U^* \mathcal{D} U = -\mathcal{D}.$$

**Lemma 5.17.** *Each noncommutative integral (55) of an element of the algebra or differential forms is (up to sign) invariant under the algebra automorphism  $\rho$  defined by*

$$\rho(a) := a, \quad \rho(a^*) := a^*, \quad \rho(b) := b^*, \quad \rho(b^*) := b. \quad (64)$$

*Proof.* For any homogeneous polynomial  $p$  and any  $k \in \mathbb{N}$ ,

$$\begin{aligned} \oint p(a, a^*, b, b^*, \mathcal{D}) \mathcal{D}^{-k} &= \oint U^* p(a, a^*, b, b^*, \mathcal{D}) \mathcal{D}^{-n} U \\ &= (-1)^k \oint p(U^* a U, U^* a^* U, U^* b U, U^* b^* U, U^* \mathcal{D} U) \mathcal{D}^{-k} \\ &= (-1)^{k+d} \oint p(\rho(a), \rho(a^*), \rho(b), \rho(b^*), \mathcal{D}) \mathcal{D}^{-k}, \end{aligned}$$

where  $d$  is the degree of  $p$  with respect to  $\mathcal{D}$ .  $\square$

**Corollary 5.18.** *For any  $n \in \mathbb{N}$ ,  $\oint (bb^*)^n b^* \delta b \mathcal{D}^{-1} = \oint (bb^*)^n b \delta b^* \mathcal{D}^{-1}$ .*

**Lemma 5.19.** For any  $x, y \in \Psi^0(\mathcal{A})$ ,

- (i)  $\oint xy|\mathcal{D}|^{-1} = \oint yx|\mathcal{D}|^{-1} + \oint x\delta(y)|\mathcal{D}|^{-2} - \oint x\delta^2(y)|\mathcal{D}|^{-2}.$
- (ii)  $\oint zx\mathcal{D}^{-1}y\mathcal{D}^{-1} = \oint zxy\mathcal{D}^{-2},$  where  $z \in \mathcal{A}$  contains  $b$  or  $b^*$ .

*Proof.* (i) is direct consequence of the trace property of  $\oint$  and the fact that  $OP^{-4}$  operators are trace-class.

ii) We calculate:

$$\begin{aligned} \oint zx\mathcal{D}^{-1}y\mathcal{D}^{-1} &= \oint zx(y\mathcal{D}^{-1} - \mathcal{D}^{-1}[\mathcal{D}, y]\mathcal{D}^{-1})\mathcal{D}^{-1} = \oint zxy\mathcal{D}^{-2} - \oint zx\mathcal{D}^{-1}[D, y]\mathcal{D}^{-2} \\ &= \oint zxy\mathcal{D}^{-2}. \end{aligned}$$

The last step is based on the observation that any integral with  $\mathcal{D}^{-3}$  vanishes if the expression integrated contains  $b$  or  $b^*$ .  $\square$

**Lemma 5.20.** For any  $n \in \mathbb{N}$ ,

- (i)  $\oint (bb^*)^n b^* db \mathcal{D}^{-1} = \frac{2}{1-q^{2n+2}}.$
- (ii)  $\oint (bb^*)^n d(bb^*) \mathcal{D}^{-1} = 0.$
- (iii)  $\oint (bb^*)^n |\mathcal{D}|^{-1} = \frac{-2(1+q^{2n})}{(1-q^{2n})^2}.$

*Proof.* (i) With  $n > 1$ , we begin with  $\oint d((bb^*)^n) \mathcal{D}^{-1} = 0$ , which follows directly from the trace property of the noncommutative integral. Expanding the expression using Leibniz rule and the commutation

$$x\mathcal{D}^{-1} = \mathcal{D}^{-1}x + \mathcal{D}^{-1}[D, x]\mathcal{D}^{-1}, \quad (65)$$

we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{n-1} \oint b^k db b^{n-k-1} (b^*)^n \mathcal{D}^{-1} + \sum_{k=0}^{n-1} \oint b^n (b^*)^k db^* (b^*)^{n-k-1} \mathcal{D}^{-1} \\ &= n \left( \oint b^{n-1} (b^*)^n db \mathcal{D}^{-1} + \oint b^n (b^*)^{n-1} db^* \mathcal{D}^{-1} \right) \\ &\quad + \sum_{k=0}^{n-1} \oint \left( b^k db \mathcal{D}^{-1} d(b^{n-k-1} (b^*)^n) \mathcal{D}^{-1} + b^n (b^*)^k db^* \mathcal{D}^{-1} d((b^*)^{n-k-1}) \mathcal{D}^{-1} \right). \end{aligned}$$

Using Lemma 5.19,

$$\begin{aligned} 0 &= n \oint (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} \\ &\quad + \oint \left( \frac{1}{2} n(n-1) b^{n-2} (b^*)^n db db + n^2 b^{n-1} (b^*)^{n-1} db db^* + \frac{1}{2} n(n-1) b^n (b^*)^{n-2} db^* db^* \right) \mathcal{D}^{-2}. \end{aligned}$$

The integrals with  $\mathcal{D}^{-2}$  could be easily calculated when we take restrict ourselves to calculations modulo ideal  $\mathcal{R}$ :

$$n \oint (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} = -2 (n(n-1) - 2n^2 + n(n-1)) \frac{1}{1-q^{2n}} = 4n \frac{1}{1-q^{2n}}.$$

Hence  $\oint (bb^*)^{n-1} (b^* db + b db^*) \mathcal{D}^{-1} = \frac{4}{1-q^{2n}}$ , which together with Corollary 5.18 proves *i*).

(ii) In a similar way,  $\oint (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1} = 0 = \oint (bb^*)^{n-1} d(aa^*) \mathcal{D}^{-1}$  implies:

$$\begin{aligned} 0 &= \sum_{k=0}^{n-1} (bb^*)^{n-k-1} d(bb^*) (bb^*)^k \mathcal{D}^{-1} \\ &= n \oint (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1} + \frac{1}{2} n(n-1) \oint (bb^*)^{n-2} d(bb^*) d(bb^*) \mathcal{D}^{-2} \\ &= n \oint (bb^*)^{n-1} d(bb^*) \mathcal{D}^{-1}, \end{aligned}$$

where in the last step we used that  $d(bb^*) \in \mathcal{R}$ . The identity (ii) now follows from the equality  $aa^* = 1 - bb^*$ .

(iii) Using Lemma 5.19, we get

$$A_n := \oint (bb^*)^n |D|^{-1} = \oint (bb^*)^n (aa^* + bb^*) |D|^{-1}$$

and we push now  $a^*$  through  $|D|^{-1}$  and from cyclicity of the trace through  $(bb^*)^n$ ,

$$= A_{n+1} + \oint (bb^*)^n q^{2n} a^* a |D|^{-1} + \oint (bb^*)^n q^{2n} a \delta(a^*) |D|^{-2}$$

the last term being calculated explicitly, since up to ideal  $\mathcal{R}$ ,  $a \delta(a^*) \simeq L_q + M_q - 1$ ,

$$= A_{n+1} (1 - q^{2n+2}) + q^{2n} A_n + 4 \left( \frac{1}{1-q^{2n+2}} - \frac{1}{1-q^{2n}} \right),$$

which leads to

$$A_n (1 - q^{2n}) + \frac{4}{1-q^{2n}} = A_{n+1} (1 - q^{2n+2}) + \frac{4}{1-q^{2n+2}}.$$

Assuming  $A_n = \frac{f_n}{(1-q^{2n})^2}$  we have  $\frac{f_{n+4}}{1-q^{2n}} = \frac{f_{n+1}+4}{1-q^{2n+2}}$ , and taking into account that  $A_0 = -2 \frac{1+q^2}{(1-q^2)^2}$ , we obtain  $A_n = -2 \frac{1+q^{2n}}{(1-q^{2n})^2}$ .  $\square$

Finally, to get Proposition 5.16, it remains to prove

**Lemma 5.21.** *For  $n \geq 1$ ,*

$$\begin{aligned} \oint (bb^*)^n a da^* \mathcal{D}^{-1} &= \frac{-2q^{4n+2} - 2q^{4n} - 2q^{2n+2} + 6q^{2n}}{(1-q^{2n})^2(1-q^{2n+2})}, \\ \oint (bb^*)^n a^* da \mathcal{D}^{-1} &= \frac{6q^{2n+2} - 2q^{2n} - 2q^2 - 2}{(1-q^{2n})^2(1-q^{2n+2})}. \end{aligned}$$

*Proof.* First, using Leibniz rule, (65) and Lemma 5.19 we have (for  $n \geq 1$ )

$$\oint (bb^*)^n a da^* \mathcal{D}^{-1} = -q^{2n} \oint (bb^*)^n a^* da - \oint (bb^*)^n da da^* \mathcal{D}^{-2}.$$

Further, we use the identity (56):

$$\oint (bb^*)^n (a^* da + q^2 a da^* + q^2 b db^* + q^2 b^* db) \mathcal{D}^{-1} = (1 - q^2) \oint (bb^*)^n |D|^{-1}.$$

taking into account that  $F\mathcal{D} = |D|$ .

These equations give together a system of linear equations

$$\begin{aligned} \oint (bb^*)^n a da^* \mathcal{D}^{-1} + q^{2n} \oint (bb^*)^n a^* da \mathcal{D}^{-1} &= -4 \left( \frac{1}{1-q^{2n+2}} - \frac{1}{1-q^{2n}} \right), \\ q^2 \oint (bb^*)^n a da^* \mathcal{D}^{-1} + \oint (bb^*)^n a^* da \mathcal{D}^{-1} &= -2(1 - q^2) \frac{1+q^{2n}}{(1-q^{2n})^2} - \frac{4q^2}{1-q^{2n+2}} \end{aligned}$$

which is solved by the expressions stated in the lemma.  $\square$

### 5.2.3 The noncommutative integrals at $|D|^{-2}$

We need to separate this task into two problems. First, we shall to calculate all integrals  $\oint x \delta(z) |D|^{-2}$ , with  $x \in \mathcal{A}(SU_q(2))$  and  $z$  being one of the generators. The second problem is to calculate  $\oint x \delta(y) \delta(z) |D|^{-2}$ , with both  $y$  and  $z$  being the generators  $\{a, a^*, b, b^*\}$ .

**Lemma 5.22.** *The only a priori non-vanishing integrals of the type  $\oint x \delta(z) |D|^{-2}$  are for  $n \in \mathbb{N}$ :*

$$\begin{aligned} \oint (bb^*)^n b^* \delta(b) |D|^{-2} &= \oint (bb^*)^n b \delta(b^*) |D|^{-2} = 0, \\ \oint (bb^*)^n a \delta(a^*) |D|^{-2} &= \frac{4q^{2n}(1-q^2)}{(q^{2n+2}-1)(1-q^{2n})}, \quad n > 0 \\ \oint (bb^*)^n a^* \delta(a) |D|^{-2} &= \frac{4(1-q^2)}{(1-q^{2n+2})(1-q^{2n})}. \end{aligned}$$

*Proof.* Since  $a\delta(a^*) \simeq L_q + M_q - 1$  and  $(bb^*)^n \simeq L_q^n + M_q^n$ , we get

$$(bb^*)^n a \delta(a^*) \simeq L_q^{n+1} + M_q^{n+1} - L_q^n - M_q^n$$

and the second result is obtained from Lemma 5.14. The other integrals are computed in a similar way.  $\square$



**Lemma 5.23.** *The only a priori non-vanishing integrals of the type  $\oint x dy dz |D|^{-2}$  are for  $n \in \mathbb{N}$ :*

$$\begin{aligned}
\oint (bb^*)^n (b^*)^2 db db |D|^{-2} &= \frac{4}{1-q^{2n+4}}, \\
\oint (bb^*)^n db db^* |D|^{-2} &= \frac{4}{1-q^{2n+2}}, \\
\oint (bb^*)^n (a^* b^*)(da db) |D|^{-2} &= 0, \\
\oint (bb^*)^n (ab^*)(da^* db) |D|^{-2} &= 0, \\
\oint (bb^*)^n (a^* b)(da db^*) |D|^{-2} &= 0, \\
\oint (bb^*)^n (ab)(da^* db^*) |D|^{-2} &= 0, \\
\oint (bb^*)^n (da da^*) |D|^{-2} &= \frac{4(q^{2n+2}-q^{2n})}{(1-q^{2n+2})(1-q^{2n})}, \quad n > 0 \\
\oint (bb^*)^n (da^* da) |D|^{-2} &= \frac{4(q^2-1)}{(1-q^{2n+2})(1-q^{2n})}.
\end{aligned}$$

*Proof.* This follows from Lemma 5.14 with the equivalences up to  $\mathcal{R}$  gathered in Lemma 5.15.  $\square$

## 6 Examples of spectral action

It is clear from Theorem 4.3 that any one-form of the form  $ada$ ,  $bdb$ ,  $adb$ ,  $a^*db$ , etc... do not contribute to the spectral action. Indeed, only the balanced parts of one-forms give a possibly nonzero term in the coefficients. Let us now give the values of the terms  $\oint A^n |D|^{-p}$  and the full  $\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$  for few examples

Table 1: Values of noncommutative integrals

$\mathbb{A}$	$\oint A  D ^{-3}$	$\oint A^2  D ^{-3}$	$\oint A^3  D ^{-3}$	$\oint A  D ^{-2}$	$\oint A^2  D ^{-2}$	$\oint A  D ^{-1}$	$\zeta_{\mathcal{D}_{\mathbb{A}}}(0)$
$a^* da$	2	2	2	$\frac{4q^2}{q^2-1}$	$\frac{4q^2(q^2+2)}{q^4-1}$	$\frac{3q^2+1}{2(q^2-1)}$	$\frac{11q^4+36q^2+13}{3(q^4-1)}$
$b^* db$	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$	$\frac{4q^2}{q^4-1}$
$ada^*$	-2	2	-2	$\frac{-4}{q^2-1}$	$\frac{4(2q^2+1)}{q^4-1}$	$\frac{q^2+3}{2(q^2-1)}$	$\frac{13q^4+36q^2+11}{3(q^4-1)}$
$bdb^*$	0	0	0	0	$\frac{-4}{q^4-1}$	$\frac{-2}{q^2-1}$	$\frac{4q^2}{q^4-1}$

1) Clearly the spectral action depends on  $q$ : for instance,

$$\mathcal{S}(\mathcal{D}_{a^* da}, \Phi, \Lambda) = 2 \Phi_3 \Lambda^3 - 8 \Phi_2 \Lambda^2 + \frac{q^2+15}{2(1-q^2)} \Phi_1 \Lambda^1 + \frac{11q^4+36q^2+13}{3(q^4-1)} \Phi(0).$$

2) Moreover, for  $B := a \delta a^*$  and  $A := B + B^*$ , we get since  $B \simeq B^* \pmod{\mathcal{R}}$ ,

$$\oint A^p |D|^{-k} = 2^p \oint B^p |D|^{-k}, \quad 1 \leq p \leq k \leq 3. \quad (66)$$

Thus the spectral action of the selfadjoint one-form  $\mathbb{A} := ada^* + (ada^*)^*$  is

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 + 16\Phi_2 \Lambda^2 + \frac{q^2-33}{2(1-q^2)} \Phi_1 \Lambda^1 + \frac{122q^4+168q^2-2}{3(q^4-1)} \Phi(0).$$

3) When  $B_n := (bb^*)^n b \delta b^*$ , then by Lemma (5.15),  $B_n \simeq B_n^*$ , so for  $A_n := B_n + B_n^*$ , the equation (66) is still valid and  $\oint B_n^p |\mathcal{D}|^{-k}$  are all zero but  $\oint B_n |\mathcal{D}|^{-1} = \frac{2}{1-q^{2n+2}}$  and  $\oint B_n^2 |\mathcal{D}|^{-2} = \frac{4}{1-q^{4n+4}}$ , so

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}_n}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \frac{8}{1+q^{2n+2}} \Phi(0). \quad (67)$$

Remark that this spectral action still exists as  $q \rightarrow 1$ !

Note however that the symmetrization process (66) is not true in general, for instance if  $B := a \delta b$  and  $A := B + B^*$ , then  $\oint A^2 |\mathcal{D}|^{-1} = \frac{8(q^4-q^2-1)}{(1-q^4)^2}$  while  $\oint B^2 |\mathcal{D}|^{-1} = 0$  or  $\oint [B, B^*] |\mathcal{D}|^{-1} = \frac{4}{1-q^4}$ .

4) The spectral action can be also independent of  $q$ : for instance, if  $\mathbb{A} = \frac{1}{1-q^2} \xi(\mathcal{D})$  is the  $q$ -dependent selfadjoint one-form given in (58), then,

$$\mathcal{S}(\mathcal{D}_{\mathbb{A}}, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - 8\Phi_2 \Lambda^2 + \frac{15}{2} \Phi_1 \Lambda^1 - \frac{13}{3}.$$

## 7 The commutative sphere $\mathbb{S}^3$

Since  $SU(2) \simeq \mathbb{S}^3$ , we get a concrete spinorial representation of the algebra  $\mathcal{A} := C^\infty(\mathbb{S}^3)$  on the same Hilbert space  $\mathcal{H}$  and same Dirac operator  $\mathcal{D}$  with (10) where  $q = 1$  which means that  $q$ -numbers are trivial:  $[\alpha] = \alpha$ . So

$$\begin{aligned} \pi(a) |j\mu n\rangle &:= \alpha_{j\mu n}^+ |j^+ \mu^+ n^+\rangle + \alpha_{j\mu n}^- |j^- \mu^+ n^+\rangle, \\ \pi(b) |j\mu n\rangle &:= \beta_{j\mu n}^+ |j^+ \mu^+ n^-\rangle + \beta_{j\mu n}^- |j^- \mu^+ n^-\rangle, \\ \pi(a^*) |j\mu n\rangle &:= \tilde{\alpha}_{j\mu n}^+ |j^+ \mu^- n^-\rangle + \tilde{\alpha}_{j\mu n}^- |j^- \mu^- n^-\rangle, \\ \pi(b^*) |j\mu n\rangle &:= \tilde{\beta}_{j\mu n}^+ |j^+ \mu^- n^+\rangle + \tilde{\beta}_{j\mu n}^- |j^- \mu^- n^+\rangle \end{aligned} \quad (68)$$

where

$$\begin{aligned} \alpha_{j\mu n}^+ &:= \sqrt{j+\mu+1} \begin{pmatrix} \frac{\sqrt{j+n+3/2}}{2j+2} & 0 \\ \frac{\sqrt{j-n+1/2}}{(2j+1)(2j+2)} & \frac{\sqrt{j+n+1/2}}{2j+1} \end{pmatrix}, \\ \alpha_{j\mu n}^- &:= \sqrt{j-\mu} \begin{pmatrix} \frac{\sqrt{j-n+1/2}}{2j+1} & -\frac{\sqrt{j+n+1/2}}{2j(2j+1)} \\ 0 & \frac{\sqrt{j-n-1/2}}{2j} \end{pmatrix}, \\ \beta_{j\mu n}^+ &:= \sqrt{j+\mu+1} \begin{pmatrix} \frac{\sqrt{j-n+3/2}}{2j+2} & 0 \\ -\frac{\sqrt{j+n+1/2}}{(2j+1)(2j+2)} & \frac{\sqrt{j-n+1/2}}{2j+1} \end{pmatrix}, \\ \beta_{j\mu n}^- &:= \sqrt{j-\mu} \begin{pmatrix} -\frac{\sqrt{j+n+1/2}}{2j+1} & -\frac{\sqrt{j-n+1/2}}{2j(2j+1)} \\ 0 & -\frac{\sqrt{j+n-1/2}}{2j} \end{pmatrix} \end{aligned}$$

with  $\tilde{\alpha}_{j\mu n}^\pm := (\alpha_{j^\mp \mu^\mp n^\mp}^\mp)^*$ ,  $\tilde{\beta}_{j\mu n}^\pm := (\beta_{j^\mp \mu^\mp n^\mp}^\mp)^*$ .

Note that the representation on the vectors  $v_{m,l}^j$  is not as convenient as in (11).

One can check that the generators  $\pi(a)$ ,  $\pi(b)$  and their adjoint commute and that  $[x, [\mathcal{D}', y]] = 0$  for any  $x, y \in \mathcal{A}$ .

## 7.1 Translation of Dirac operator

In general the Dirac operator is defined in a more symmetric way than that we did. So, although not absolutely necessary here, we define for the interested reader the unbounded self-adjoint translated operator  $\mathcal{D}'$  on  $\mathcal{H}$  by the constant  $\lambda$  as

$$\mathcal{D}' := \mathcal{D} + \lambda.$$

For instance, this gives for  $\lambda = -\frac{1}{2}$  in the case of  $\mathbb{S}^3$ , see [32],  $\mathcal{D}' v_{m,l}^j = (2j+1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_{m,l}^j$  so  $v_{m,l}^j$  is an eigenvector of  $|\mathcal{D}'|$ .

As the following lemma shows, the computation of noncommutative integrals involving  $\mathcal{D}$  can be reduced to the computation of certain noncommutative integrals involving  $\mathcal{D}'$ :

**Lemma 7.1.** *If  $\oint' T := \text{Res}_{s=0} \text{Tr} (T|\mathcal{D}'|^{-s})$ , then for any 1-form  $A$  on a spectral triple of dimension  $n$ ,*

$$\begin{aligned} \oint A |\mathcal{D}|^{-(n-2)} &= \oint' A |\mathcal{D}'|^{-(n-2)} + \lambda(n-2) \oint' A \mathcal{D}' |\mathcal{D}'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' A |\mathcal{D}'|^{-n}, \\ \oint A \mathcal{D}^{-(n-2)} &= \oint' A \mathcal{D}'^{-(n-2)} + \lambda(n-2) \oint' A \mathcal{D}'^{-(n-1)} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' A \mathcal{D}'^{-n}. \end{aligned}$$

*Proof.* Recall from [22, Proposition 4.8] that for any pseudodifferential operator  $P$ ,

$$\oint P |\mathcal{D}|^{-r} = \text{Res}_{s=0} \text{Tr} (P |\mathcal{D}|^{-r} |\mathcal{D}'|^{-s}).$$

Moreover, by [22, Lemma 4.3], for any  $s \in \mathbb{C}$  and  $N \in \mathbb{N}^*$

$$|\mathcal{D}|^{-s} = |\mathcal{D}'|^{-s} + \sum_{p=1}^N K_{p,s} Y^p |\mathcal{D}'|^{-s} \mod OP^{-N-1-\Re(s)} \quad (69)$$

where  $Y = \sum_{k=1}^N \frac{(-1)^{k+1}}{k} (-2\lambda \mathcal{D}' + \lambda^2)^k \mathcal{D}'^{-2k} \mod OP^{-N-1}$  and  $K_{p,s}$  are complex numbers that can be explicitly computed. Precisely, we find  $K_{p,s} = (-\frac{s}{2})^p V(p)$  where  $V(p)$  is the volume of the  $p$ -simplex. Since the spectral dimension is  $n$ , we work modulo  $OP^{-(n+1)}$ , and for  $s = n-2$ , we get from (69):  $|\mathcal{D}|^{-(n-2)} = |\mathcal{D}'|^{-(n-2)} + \lambda(n-2) \mathcal{D}' |\mathcal{D}'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} |\mathcal{D}'|^{-n} \mod OP^{-(n+1)}$ . As a consequence, we have for  $P \in OP^0$  (the  $OP^0$  spaces are the same for  $\mathcal{D}$  or  $\mathcal{D}'$ ),

$$\oint P |\mathcal{D}|^{-(n-2)} = \oint' P |\mathcal{D}'|^{-(n-2)} + \lambda(n-2) \oint' P \mathcal{D}' |\mathcal{D}'|^{-n} + \lambda^2 \frac{(n-1)(n-2)}{2} \oint' P |\mathcal{D}'|^{-n}.$$

Since  $A$  and  $AF$  are in  $OP^0$ , we get both formulae.  $\square$

## 7.2 Tadpole and spectral action on $\mathbb{S}^3$

We consider now the commutative spectral triple  $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$ . It is 1-summable since  $\langle j\mu n s | [F, \pi(x)] | j\mu n s \rangle = 0$  when  $x = a, a^*, b, b^*$  for any  $j, \mu, n, s = \uparrow, \downarrow$ .

All integrals of above lemma are zero for  $\mathbb{S}^3$ :

**Proposition 7.2.** *There is no tadpole of any order on the commutative real spectral triple  $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$ . In fact, for any one-form or  $\delta$ -one-form  $A$ ,  $\oint A \mathcal{D}^{-p} = 0$  for  $p \in \{1, 2, 3\}$ .*

*Proof.* We first want to prove  $\oint AD^{-p} = 0$  for  $p \in \{1, 2, 3\}$  and any one-form  $A$ . Since the representation is real, that is any matrix elements of the generators are real, so must be the trace of  $AD^{-p}$ . Hence  $\oint AD^{-p} = \oint A^* \mathcal{D}^{-p}$ .

The reality operator  $J$  introduced in (29) satisfies, when  $q = 1$ , the commutative relation  $JxJ^{-1} = x^*$  for  $x \in \mathcal{A}$ . Thus  $JAJ^{-1} = -A^*$ , so  $\oint AD^{-p} = \oint J(A^* \mathcal{D}^{-p})J^{-1} = -\oint A^* \mathcal{D}^{-p}$  and  $\oint AD^{-p} = 0$ . Similar proof for the  $\delta$ -form  $AF$ .  $\square$

For any selfadjoint one-form  $A$ ,  $\mathcal{D}_A := \mathcal{D} + \tilde{A} = \mathcal{D}$ . Thus, the spectral action for the real spectral triple  $(C^\infty(\mathbb{S}^3), \mathcal{H}, \mathcal{D})$  for  $\mathcal{D}_A$  is trivialized by

$$\mathcal{S}(\mathcal{D}_A, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1}). \quad (70)$$

But it is more natural to compare with the spectral action of  $\mathcal{D} + A$ . This is obtained respectively from Lemma 4.18 and general heat kernel approach [27]:

$$\mathcal{S}(\mathcal{D} + A, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 + \oint |\mathcal{D} + A|^{-1} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1})$$

since all terms of (2) in  $\Lambda^{n-k}$  are zero for  $k$  odd and  $\zeta_{\mathcal{D}+A}(0) = 0$  when  $n$  is odd: as a verification,  $\oint |\mathcal{D} + A|^{-2}$  is zero according to [22, Lemma 4.10], Lemmas 4.18 and Proposition 7.2. Similarly,  $\zeta_{\mathcal{D}+A}(0) = 0$  because in (3), all terms with  $k$  odd are zero (same proof as in Proposition 7.2) but for  $k$  even, it is not that easy to show that  $\oint AD^{-1}AD^{-1} = 0$ .

Moreover, the curvature term does not depend on  $A$ :

**Lemma 7.3.** *For any one-form  $A$  on a commutative spectral triple of dimension  $n$  based on a compact Riemannian spin<sup>c</sup> manifold without boundary, we have*

$$\oint |\mathcal{D} + A|^{-(n-2)} = \oint |\mathcal{D}|^{-(n-2)}. \quad (71)$$

*Proof.* Follows from [23, first formula page 511] with  $\rho := A = A^*$ ,  $N(\rho) = \rho$  (the constraint  $J\rho J^{-1} = \pm\rho$  is not used.)

One can also use [15, Proposition 1.149].  $\square$

From [22, Lemma 4.10]  $\oint |\mathcal{D} + A|^{-(n-2)} = \oint |\mathcal{D}|^{-(n-2)} + \frac{n(n-2)}{4} \oint (AF)^2 |\mathcal{D}|^{-3} + \frac{(n-2)^2}{4} \oint A^2 |\mathcal{D}|^{-3}$  using  $X := AD + DA + A^2$  and  $[|\mathcal{D}|, A] \in OP^0$ , but again, it is not that easy to show that the last two terms cancelled: for instance here, for  $B = b[\mathcal{D}, b^*]$ , we obtain by direct computation (using the easiest translated Dirac operator  $\mathcal{D}'$ )

$$\begin{aligned} \text{Tr}(B^2 |\mathcal{D}'|^{-3-s}) &= \text{Tr}((B^*)^2 |\mathcal{D}'|^{-3-s}) = \frac{1}{2} \text{Tr}(BB^* |\mathcal{D}'|^{-3-s}) = \frac{1}{2} \text{Tr}(B^*B |\mathcal{D}'|^{-3-s}) \\ &= \frac{4}{3} \sum_{2j \in \mathbb{N}} \frac{j+1}{(2j+1)^{2+s}}, \end{aligned}$$

so  $\oint B^2 |\mathcal{D}'|^{-3} = \frac{2}{3}$ . Similarly, one checks that  $\oint (BF)^2 |\mathcal{D}|^{-3} = \frac{1}{2} \oint BFB^*F |\mathcal{D}|^{-3} = -\frac{2}{9}$ . Thus if  $A := B + B^*$ ,  $\oint A^2 |\mathcal{D}|^{-3} = \oint A^2 |\mathcal{D}'|^{-3} = 4$  and  $\oint (AF)^2 |\mathcal{D}|^{-3} = -\frac{4}{3}$  which yields to (71).

Thus for any one-form  $A$  on the 3-sphere,

$$\mathcal{S}(\mathcal{D} + A, \Phi, \Lambda) = 2\Phi_3 \Lambda^3 - \frac{1}{2} \Phi_1 \Lambda^1 + \mathcal{O}(\Lambda^{-1}, A)$$

which as (70) is not identical to (67) which contains a nonzero constant term  $\Lambda^0$  for  $q = 1$ .

## 8 Conclusion

We computed in this paper the full spectral action on the  $SU_q(2)$  spectral triple of [21] with the reality operator  $J$  (notice the change of definition for pseudodifferential operators.) The dimension spectrum being a finite set, there is only a finite number of terms in the spectral action expansion. The tadpole hypothesis is not satisfied on  $SU_q(2)$ . We saw that the action depends on  $q$  and the limit  $q \rightarrow 1$  does not exist automatically. When it exists, such limit does not lead to the associated action on the commutative sphere  $\mathbb{S}^3$ . The sign  $F$  of the Dirac operator has special properties: first, it commutes modulo  $OP^{-\infty}$  with elements of the algebra, and second, it can be seen as a one-form, giving terms independent of  $q$  in the spectral action. Here, we were interested in the computation of the spectral action of a quantum group. Naturally, it would be interesting to investigate other related cases like the Podleś spheres [17, 19] or the Euclidean quantum spheres [20, 35], especially the 4-sphere [18].

## References

- [1] C. Bär, “The Dirac operator on homogeneous spaces and its spectrum on 3-dimensional lens spaces”, Arch. Math. **59** (1992), 65–79.
- [2] P.N. Bibikov and P.P. Kulish, “Dirac operators on the quantum group  $SU_q(2)$  and the quantum sphere”, Zap. Nauchn. Sem. St. Petersburg. Otdel. Mat. Inst. Steklov. **245** (1997), 49, Vopr. Kvant. Teor. Polya i Stat. fiz. **14** (1997), 49–65; translated in J. Math. Sci. **100** (2000), 2039–2050.
- [3] L. Carminati, B. Iochum and T. Schücker, “Noncommutative Yang-Mills and noncommutative relativity: a bridge over troubled water”, Eur. Phys. J. **C 8** (1999), 697–709.
- [4] P. S. Chakraborty and A. Pal, “Equivariant spectral triples on the quantum  $SU(2)$  group”, K-Theory **28** (2003), 107–126.
- [5] P. S. Chakraborty and A. Pal, “On equivariant Dirac operator for  $SU_q(2)$ ”, Proc. Indian Acad. Sci. **116** (2003), 531–541.
- [6] P. S. Chakraborty and A. Pal, “Spectral triples and associated Connes-de Rham complex for the quantum  $SU(2)$  and the quantum sphere,” Commun. Math. Phys. **240** (2003), 447–456.
- [7] A. Chamseddine and A. Connes, “The spectral action principle”, Commun. Math. Phys. **186** (1997), 731–750.
- [8] A. Chamseddine and A. Connes, “Inner fluctuations of the spectral action”, J. Geom. Phys. **57** (2006), 1–21.
- [9] A. Chamseddine, A. Connes and M. Marcolli, “Gravity and the standard model with neutrino mixing”, [arXiv:hep-th/0610241].
- [10] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego, 1994.
- [11] A. Connes, “Geometry from the spectral point of view”, Lett. Math. Phys. **34** (1995), 203–238.

- [12] A. Connes, “Noncommutative geometry and reality”, J. Math. Phys. **36** (1995), 6194–6231.
- [13] A. Connes, “Cyclic cohomology, quantum group symmetries and the local index formula for  $SU_q(2)$ ”, J. Inst. Math. Jussieu **3** (2004), 17–68.
- [14] A. Connes and G. Landi, “Noncommutative manifolds, the instanton algebra and isospectral deformations”, Commun. Math. Phys. **221** (2001), 141–159.
- [15] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, to appear.
- [16] A. Connes and H. Moscovici, “The local index formula in noncommutative geometry”, Geom. Funct. Anal. **5** (1995), 174–243.
- [17] F. D’Andrea and L. Dąbrowski, ”Local index formula on the equatorial Podleś sphere”, Lett. Math. Phys. **75** (2006), 235–254.
- [18] F. D’Andrea, L. Dąbrowski and G. Landi, ”The isospectral Dirac operator on the 4-dimensional quantum Euclidean sphere”, arXiv:math/0611100.
- [19] F. D’andrea, L. Dąbrowki, G. Landi and E. Wagner, ”Dirac operators on all Podleś spheres”, J. Noncommut. Geom. **1** (2007), 213–239.
- [20] L. Dąbrowski, ”Geometry of quantum spheres”, J. Geom. Phys. **56** (2005), 86–107.
- [21] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J. Várilly, “The Dirac operator on  $SU_q(2)$ ”, Commun. Math. Phys. **259** (2005), 729–759.
- [22] D. Essouabri, B. Iochum, C. Levy and A. Sitarz, “Spectral action on noncommutative torus”, J. Noncommut. Geom. **2** (2008), 53–123.
- [23] R. Estrada, J. M. Gracia-Bondía and J. C. Várilly, “On summability of distributions and spectral geometry”, Commun. Math. Phys. **191** (1998), 219–248.
- [24] V. Gayral and B. Iochum, “The spectral action for Moyal plane”, J. Math. Phys. **46** (2005), no. 4, 043503, 17 pp.
- [25] V. Gayral, B. Iochum and J. C. Várilly, “Dixmier traces on noncompact isospectral deformations”, J. Funct. Anal. **237** (2006), 507–539.
- [26] V. Gayral, B. Iochum and D. V. Vassilevich, “Heat kernel and number theory on NC-torus”, Commun. Math. Phys. **273** (2007), 415–443.
- [27] P. B. Gilkey, *Asymptotic Formulae in Spectral Geometry*, Chapman & Hall/CRC, Boca Raton, FL, 2004.
- [28] D. Goswami, “Some noncommutative geometric aspects of  $SU_q(2)$ ”, math-ph/018003.
- [29] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, Birkhäuser, Boston, 2001.
- [30] H. Grosse and R. Wulkenhaar, ”8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory”, arXiv:0709.0095.

- [31] E. Hawkins and G. Landi, “Fredholm modules for quantum Euclidean spheres”, J. Geom. Phys. **49** (2004), 272–293.
- [32] Y. Homma, “A representation of  $Spin(4)$  on the eigenspinors of the Dirac operator on  $S^3$ ”, Tokyo J. Math. **23** (2000), 453–472.
- [33] A. Klimyk and K. Schmüdgen, *Quantum Groups and Their Representations*, Text and Monographs in Physics, Springer-Verlag, Berlin, 1997.
- [34] M. Knecht and T. Schücker, “Spectral action and big desert”, Phys. Lett. B **640** (2006), 272–277.
- [35] G. Landi, “Noncommutative spheres and instantons”, in *Quantum field theory and non-commutative geometry*, U. Carow-Watamura, Y. Maeda, S. Watamura, Lecture Notes in Physics, Springer, 2005, 3–56.
- [36] K. Schmüdgen, “Commutator representations of differential calculi on the quantum group  $SU_q(2)$ ”, J. Geom. Phys. **31** (1999), 241–264.
- [37] W. van Suijlekom, *The Geometry of Noncommutative Spheres and their Symmetries*, PhD thesis, Trieste 2005.
- [38] W. van Suijlekom, L. Dąbrowski, G. Landi, A. Sitarz and J. C. Várilly, “The local index formula for  $SU_q(2)$ ”, K-Theory **35** (2005), 375–394.
- [39] S. Woronowicz, “Twisted  $SU(2)$  group. An example of a non-commutative differential calculus”, Publ. RIMS, Kyoto Univ. **23** (1987), 117–181.